Application of Integrals in Dynamics (PHY 2400; ME2210; BME3212)

\[ x(t) \rightarrow \mathbf{v}, \mathbf{a} \]

- \( x(t) \) ... position (m)
- \( v(t) = \frac{dx}{dt} \) ... velocity (m/s)
- \( a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} \) ... acceleration (m/s²)

Given \( a(t) \), both \( v(t) \) and \( x(t) \) can be determined by integration:

\[ \frac{dv}{dt} = a(t) \]

Integrating both sides between \( t = 0 \) and any time \( t \):

\[ \int_0^t \frac{dv}{dt} dt = \int_0^t a(t) dt \]

By definition, \( \int \frac{dv}{dt} dt = v(t) ! \)

\[ \Rightarrow [v(t)]_0^t = \int_0^t a(t) dt \]

\[ v(t) - v(0) = \int_0^t a(t) dt \]

\[ \Rightarrow \boxed{v(t) = \int_0^t a(t) dt + v(0)} \]

\( v(0) \) ... initial velocity

Given \( v(t) \), \( x(t) \) can be determined by integration:

\[ \frac{dx}{dt} = v(t) \]

Integrating both sides between \( t = 0 \) and any time \( t \),

\[ \int_0^t \frac{dx}{dt} dt = \int_0^t v(t) dt \]
\[ \frac{dx}{dt} \, dt = v(t) \, dt \]

by definition, \( \int \frac{dx}{dt} \, dt = x(t) \):

\[ \Rightarrow \quad [x(t)]_0^t = \int_0^t v(t) \, dt \]

\[ x(t) - x(0) = \int_0^t v(t) \, dt \]

\[ x(t) = \int_0^t v(t) \, dt + x(0) \]

**EXAMPLE:**

A ball is dropped from a height of 1.0 m at \( t = 0 \) seconds:

\[ y(0) = 0 \quad \text{at} \quad t = 0 \quad \text{m} \]

\[ 1.0 \text{m} \]

\[ a = g = -9.81 \text{ m/s}^2 \]

Find: \( v(t) \), \( y(t) \), and time to impact:

**Solution:**

\[ \frac{dv}{dt} = at(t) = -9.81 \]

Integrating both sides,

\[ \int_0^t \frac{dv}{dt} \, dt = \int_0^t -9.81 \, dt \]

\[ [v(t)]_0^t = [-9.81t]_0^t \]

\[ v(t) - v(0) = -9.81t - 0 \]

\[ v(t) = -9.81t + v(0) \]

*Here \( v(0) = 0 \) (dropped from rest)*

\[ \therefore v(t) = -9.81t \, \text{m/s} \]
Given \( v(t) \), we can integrate again to get \( y(t) \):

\[
\frac{dy}{dt} = v(t)
\]

Integrating both sides,

\[
\int_0^t \frac{dy}{dt} \, dt = \int_0^t v(t) \, dt
\]

\[
[y(t)]_0^t = \int_0^t -9.81t \, dt
\]

\[
y(t) - y(0) = \left[-\frac{9.81}{2} t^2\right]_0^t
\]

\[
y(t) = -4.905t^2 - y(0) = y(0) - 4.905t^2
\]

Here, \( y(0) = 1.0 \text{ m} \) (initial height)

\[
y(t) = 1.0 - 4.905t^2 \text{ m}
\]

Impact: when \( y(t) = 1.0 - 4.905t^2 = 0 \)

\[
0 = 4.905t^2 - 1.0
\]

\[
t = \sqrt{\frac{1.0}{4.905}} = 0.452 \text{ s}
\]
**Example:** Suppose the ball is thrown upwards with an initial velocity \( v(0) = v_0 = 4.43 \text{ m/s} \)

\[
\begin{align*}
y(t) & \uparrow \quad v(0) = 4.43 \text{ m/s} \\
\downarrow & \quad t = 0 \\
\circ & \quad a = g = -9.81 \text{ m/s}^2
\end{align*}
\]

Find: \( v(t) \) and \( y(t) \)

**Solution:** \( a = \frac{dv}{dt} = -9.81 \)

Integrating,

\[
\int_0^t \frac{dv}{dt} \, dt = \int_0^t -9.81 \, dt
\]

\[
[v(t)]_0^t = [-9.81t]_0^t
\]

\[
v(t) - v(0) = -9.81t - 0
\]

Here, \( v(0) = 4.43 \text{ m/s} \)

\[
\Rightarrow \quad v(t) = 4.43 - 9.81t \text{ m/s}
\]

Given \( v(t) \), integrate to get \( y(t) \):

\[
\frac{dy}{dt} = v(t) = 4.43 - 9.81t
\]

Integrating,

\[
\int_0^t \frac{dy}{dt} \, dt = \int_0^t 4.43 - 9.81t \, dt
\]

\[
[y(t)]_0^t = [4.43t - \frac{9.81}{2}t^2]_0^t
\]

\[
y(t) - y(0) = 4.43t - 4.905t
\]

Here \( y(0) = 0 \), \[
y(t) = 4.43t - 4.905t^2 \text{ m}
\]
Graphical Interpretation:

The velocity $v(t)$ can be determined as the area under the graph of $a(t)$:

$$ A = \int_{t_1}^{t_2} a(t) \, dt $$

By definition, $a(t) = \frac{dv}{dt}$.

Integrating both sides between $t_1$ and $t_2$,

$$ \int_{t_1}^{t_2} a(t) \, dt = \int_{t_1}^{t_2} \frac{dv}{dt} \, dt $$

$$ \int_{t_1}^{t_2} a(t) \, dt = \left[ v(t) \right]_{t_1}^{t_2} $$

$$ \int_{t_1}^{t_2} a(t) \, dt = v(t_2) - v(t_1) = V_2 - V_1 $$

area under $a(t)$ between $t_1$ and $t_2$  

change in $v(t)$ between $t_1$ and $t_2$
Similarly, the position $x(t)$ can be determined as the area under the velocity curve $v(t)$:

$$A = \int_{t_1}^{t_2} v(t) \, dt$$

By definition, $v(t) = \frac{dx}{dt}$

Integrating both sides between $t_1$ and $t_2$,

$$\int_{t_1}^{t_2} v(t) \, dt = \int_{t_1}^{t_2} \frac{dx}{dt} \, dt$$

$$\int_{t_1}^{t_2} v(t) \, dt = [x(t)]_{t_1}^{t_2}$$

$$\int_{t_1}^{t_2} v(t) \, dt = x(t_2) - x(t_1) = x_2 - x_1$$

area under $v(t)$

change in $x(t)$

between $t_1$ and $t_2$

between $t_1$ and $t_2$
EXAMPLE: the acceleration of a particle is measured as:

\[
\begin{array}{c}
\text{a(t)} \\
(\text{m/s}^2)
\end{array}
\]

\[
\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
10
\end{array}
\]

Knowing the particle starts from rest, sketch the velocity \( v(t) \) and the position \( x(t) \) using integrals.

**Solution:** begin with \( v(t) \), knowing \( v(0) = 0 \) and \( v(t) = \int a(t) \, dt \)

\[0 \leq t \leq 2 \text{ s: } a(t) = 2 \text{ (constant)} \]

\[\Rightarrow v(t) = \int a(t) \, dt \text{ (straight line)}\]

Also,

\[V_2 - V_0 = \int_0^2 a(t) \, dt \quad \Rightarrow \quad V_2 = 0 + 4 = 4 \frac{m}{s}\]

\[2 \leq t \leq 4 \text{ s: } a(t) = 0 \quad \Rightarrow \quad v(t) = \int a(t) \, dt \text{ (constant)} \]

\[V_4 - V_2 = \int_2^4 a(t) \, dt \quad \Rightarrow \quad V_4 = V_2 = 4 \frac{m}{s}\]

\[4 \leq t \leq 6 \text{ s: } a(t) = -2 \quad \text{(constant)} \]

\[v(t) = \int a(t) \, dt \text{ (straight line)}\]

\[V_6 - V_4 = \int_4^6 a(t) \, dt = (-2)(2) = -4\]

\[V_6 = V_4 - 4 \quad \Rightarrow \quad V_6 = 4 - 4 = 0 \frac{m}{s}\]
Now we use $v(t)$ to sketch $x(t)$ knowing $x(t) = \int v(t) \, dt$

$0 \leq t \leq 2 \, s$: $v(t)$ is linear, $x(t) = \int v(t) \, dt$ (quadratic)

also, $x_2 - x_0 = \int_0^2 v(t) \, dt = \frac{1}{2}(2)(4) = 4$

area under $v(t)$

$x_2 = x_0 + 4 = 0 + 4 = 4 \, m$

$2 \leq t \leq 4 \, s$: $v(t)$ (constant), $x(t) = \int v(t) \, dt$ (linear)

also, $x_4 - x_2 = \int_2^4 v(t) \, dt = (2)(4) = 8$

$x_4 = x_2 + 8 = 8 + 4 = 12 \, m$

$4 \leq t \leq 6 \, s$: $v(t)$ (linear), $x(t) = \int v(t) \, dt$ (quadratic)

also, $x_6 - x_4 = \int_4^6 v(t) \, dt = \frac{1}{2}(2)(4) = 4$

$x_6 = 4 + x_4 = 4 + 12 = 16 \, m$