Application of Derivatives in Dynamics:

Position, Velocity, and Acceleration (PHY 2400 : ME 2210)

→ The motion of an object is defined by its position, \( x(t) \):

\[
\begin{align*}
\frac{dx}{dt} & = v(t) \\
\frac{d^2x}{dt^2} & = a(t)
\end{align*}
\]

→ The velocity \( v(t) \) is the instantaneous rate of change of the position (i.e., the derivatives):

\[
v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}
\]

(slope of \( x(t) \))

→ The acceleration \( a(t) \) is the instantaneous rate of change of the velocity:

\[
a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}
\]

(slope of \( v(t) \))
EXAMPLE: Velocity \ Acceleration: \ (PHY\ 2400 \ ; \ ME\ 2210)

The motion of a particle is defined by its position, \( x(t) \) in m:

\[
\begin{align*}
\begin{array}{c}
\text{Position} \\
\downarrow \\
\text{Velocity} \ \\
\downarrow \\
\text{Acceleration}
\end{array}
\end{align*}
\]

Determine the position, velocity, and acceleration at \( t = 0.5 \) s for:

(a) \( x(t) = \sin 2\pi t \) m

(b) \( x(t) = 3t^3 - 4t^2 + 2t + 6 \) m

(c) \( x(t) = 20 \cos(3\pi t) - 5t^2 \) m

Solution:

(a) \( x(t) = \sin 2\pi t \) m

\[
\begin{align*}
\mathbf{v}(t) &= \frac{dx}{dt} = \frac{d}{dt}(\sin 2\pi t) = 2\pi \cos(2\pi t) \text{ m/s} \\
\mathbf{a}(t) &= \frac{dv}{dt} = \frac{d}{dt}(2\pi \cos 2\pi t) = -4\pi^2 \sin 2\pi t \text{ m/s}^2
\end{align*}
\]

Now evaluating at \( t = 0.5 \) s:

\[
\begin{align*}
x(0.5) &= \sin(2\pi (0.5)) = \sin\pi = 0 \text{ m} = x(0.5) \\
\mathbf{v}(0.5) &= 2\pi \cos(2\pi (0.5)) = 2\pi \cos\pi = -2\pi \text{ m/s} = \mathbf{v}(0.5) \\
\mathbf{a}(0.5) &= -4\pi^2 \sin(2\pi (0.5)) = -4\pi^2 \sin\pi = 0 \text{ m/s}^2 = \mathbf{a}(0.5)
\end{align*}
\]

(b) \( x(t) = 3t^3 - 4t^2 + 2t + 6 \) m

\[
\begin{align*}
\mathbf{v}(t) &= \frac{dx}{dt} = \frac{d}{dt}(3t^3 - 4t^2 + 2t + 6) \\
&= 9t^2 - 8t + 2 \text{ m/s} \\
\mathbf{a}(t) &= \frac{dv}{dt} = \frac{d}{dt}(9t^2 - 8t + 2) = 18t - 8 \text{ m/s}^2
\end{align*}
\]
Again, evaluating at $t = 0.5 \text{s}$:

\[ x(0.5) = 3(0.5)^3 - 4(0.5)^2 + 2(0.5) + 6 = 0.375 \text{ m} = x(0.5) \]
\[ v(0.5) = 9(0.5)^2 - 8(0.5) + 2 = 0.25 \frac{\text{m}}{\text{s}} = v(0.5) \]
\[ a(0.5) = 18(0.5) - 8 = 1.0 \frac{\text{m}}{\text{s}^2} = a(0.5) \]

(c) \[ x(t) = 20 \cos 3\pi t - 5t^2 \text{ m} \]
\[ v(t) = \frac{dx}{dt} = \frac{d}{dt}(20 \cos 3\pi t - 5t^2) = -60\pi \sin 3\pi t - 10t \frac{\text{m}}{\text{s}} \]
\[ a(t) = \frac{dv}{dt} = \frac{d}{dt}(-60\pi \sin 3\pi t - 10t) = -180\pi^2 \cos 3\pi t - 10 \frac{\text{m}}{\text{s}^2} \]

Evaluating at $t = 0.5 \text{s}$:

\[ x(0.5) = 20 \cos (3\pi (0.5)) - 5(0.5)^2 = -1.25 \text{ m} = x(0.5) \]
\[ v(0.5) = -60\pi \sin (3\pi (0.5)) - 10(0.5) = 183.5 \frac{\text{m}}{\text{s}} = v(0.5) \]
\[ a(0.5) = -180\pi^2 \cos (3\pi (0.5)) - 10 = -10 \frac{\text{m}}{\text{s}^2} = a(0.5) \]
EXAMPLE: Velocity: Acceleration: (PHY 2400; ME 2210)

The motion of a particle in the vertical plane is defined by its position, \( y(t) \) in meters:

\[
\begin{align*}
\uparrow & \quad \Theta \quad \downarrow \\
\vec{v} & \quad \vec{a} \\
y(t) & \\
& \quad \frac{1}{3} t^3 - 5t^2 + 21t + 10 \text{ m}
\end{align*}
\]

Determine the values of the position and acceleration when the velocity is zero.

Solution: \( y(t) = \frac{1}{3} t^3 - 5t^2 + 21t + 10 \text{ m} \)

\[
v(t) = \frac{dy}{dt} = t^2 - 10t + 21 \text{ m/s}
\]

\[
a(t) = \frac{dv}{dt} = \frac{d^2 y}{dt^2} = 2t - 10 \text{ m/s}^2
\]

Zero velocity: \( v(t) = t^2 - 10t + 21 = 0 \)

Solving via factoring: \( v(t) = (t - 3)(t + 7) = 0 \)

\[
\Rightarrow t = 3 \text{ s}; \quad t = 7 \text{ s}
\]

*NOTE: Could also use quadratic formula:

\[
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{16}}{2} = 5 \pm 2
\]

\[
\therefore t = 3 \text{ s}; \quad t = 7 \text{ s}
\]
\( t = 3 : \ y(3) = \frac{1}{3}(3)^3 - 5(3)^2 + 21(3) + 10 = 37 \text{ m} = y(3) \)
\( a(3) = 2(3) - 10 = -4 \text{ m/s}^2 = a(3) \)
\( t = 7 : \ y(7) = \frac{1}{3}(7)^3 - 5(7)^2 + 21(7) + 10 = 26.3 \text{ m} = y(7) \)
\( a(7) = 2(7) - 10 = 4 \text{ m/s}^2 = a(7) \)

**NOTE:** above information can be used to sketch a graph of \( y(t) \):

\[ y(t) = \frac{1}{3}t^3 - 5t^2 + 21t + 10 \text{ m} \]

- at \( t = 0 \), \( y = 10 \text{ m} \)
- at \( t = 3 \), \( y = 37 \text{ m} \), \( v = \frac{dy}{dt} = 0 \text{ m/s} \), \( a = \frac{dv}{dt} = -4 \text{ m/s}^2 < 0 \)
  \( \therefore \) at \( t = 3 \), we have a local **maximum** \( \bigcirc \)
- at \( t = 7 \), \( y = 26.3 \text{ m} \), \( v = \frac{dy}{dt} = 0 \text{ m/s} \), \( a = \frac{dv}{dt} = 4 \text{ m/s}^2 > 0 \)
  \( \therefore \) at \( t = 7 \), we have a local **minimum** \( \bigcirc \)
Example: Velocity : Acceleration: (PHY2400 : ME2210)

The acceleration of a particle is measured as follows:

Knowing that the particle starts from rest and travels a total of 16 m, sketch graphs of the position $x(t)$ and velocity $v(t)$.

Solution: Begin with $v(t)$, knowing that $v(0) = 0$ and the slope of $v(t)$ is $a(t)$: $\frac{dv}{dt} = a(t)$

$0 \leq t \leq 2\ s$: $a(t) = \frac{dv}{dt} = 2$ (constant slope)

$2 \leq t \leq 4\ s$: $a(t) = \frac{dv}{dt} = 0$ ($v(t)$ is constant)

$4 \leq t \leq 6\ s$: $a(t) = \frac{dv}{dt} = -2$ (constant slope)

Now we can use graph of $v(t)$ to construct $x(t)$ knowing that the slope of $x(t)$ is $v(t)$: $\frac{dx}{dt} = v(t)$. 
\(0 \leq t \leq 2\text{ s}: \ v(t) \text{ is a straight line}; \ v(t) = \frac{dx}{dt} = 2t\)

from the derivative handout, \(x(t) = t^2 + C\)
(i.e. \(\frac{dx}{dt} = \frac{d}{dt}(t^2 + C) = 2t \checkmark\))

since \(x(0) = 0\), \(C = 0\)
\[\Rightarrow x(t) = t^2, \quad x(0) = 0, \quad x(2) = 4\text{ m}\]

\(2 \leq t \leq 4\text{ s}: \ v(t) = \frac{dx}{dt} = 4\ \ (\text{constant slope})\)
\[\Rightarrow x(t) \text{ is a straight line w/ slope of 4}\]

\(4 \leq t \leq 6\text{ s}: \ v(t) \text{ is a straight line that is decreasing}\)
\[\Rightarrow x(t) \text{ is a quadratic w/ decreasing slope, and}\]
\[\text{ending at } x = 16\text{ m (given) w/ zero slope (from } v(t))\]
EXAMPLE: Velocity & Acceleration (ME4140, ME4600)

A mass $m$ impacts a cantilever with velocity $V_0$:

\[
\begin{align*}
E I & \quad \ell \quad \text{length (m)} \\
V_0 (\text{m/s}) & \quad \ell \\
\downarrow & \\
m \quad y(t)
\end{align*}
\]

$E I$ ... flexural rigidity (N·m²)

The resulting displacement of the cantilever is $y(t)$,

\[
y(t) = \frac{V_0}{\omega} \sin \omega t, \quad \text{where} \quad \omega = \sqrt{\frac{3EI}{m\ell^3}}
\]

Find the following:

(a) the maximum displacement, $y_{\text{max}}$

(b) the values of the displacement and acceleration when the velocity is zero

Solution:

(a) $y_{\text{max}}$ occurs when $\sin \omega t = 1$

\[
\Rightarrow y_{\text{max}} = \frac{V_0}{\omega} = \sqrt{\frac{3EI}{m\ell^3}} = V_0 \sqrt{\frac{m\ell^3}{3EI}} \quad \text{m}
\]

(b) velocity, $v(t) = \frac{dy}{dt} = \frac{d}{dt} \left( \frac{V_0}{\omega} \sin \omega t \right)$

\[
\therefore v(t) = \frac{V_0}{\omega} \omega \cos \omega t = V_0 \cos \omega t \quad (\text{m/s})
\]

velocity equal to zero: $v(t) = 0 = V_0 \cos \omega t$

occurs when $\Rightarrow \omega t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$
Selecting, \( \omega t = \frac{\pi}{2} \): 

\[
y\left(\frac{\pi}{2}\right) = \frac{V_0}{c_0} \sin \left(\frac{\pi}{2}\right) = \frac{V_0}{c_0} = y_{\text{max}}!!
\]

\[\Rightarrow \text{NOTE: } y(t) \text{ is maximum when } v(t) = \frac{dy}{dt} = 0\]

acceleration: \( a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2} \)

\[
a(t) = \frac{d}{dt}(V_0 \cos \omega t) = -V_0 \omega \sin \omega t
\]

when \( v(t) = 0 \), \( \omega t = \frac{\pi}{2} \),

\[
a(t) = -V_0 \omega \sin \omega t \Rightarrow a\left(\frac{\pi}{2}\right) = -V_0 \omega \sin \left(\frac{\pi}{2}\right)
\]

\[
a\left(\frac{\pi}{2}\right) = -V_0 \omega = -V_0 \sqrt{\frac{3EI}{\pi I}} \text{ m/s}^2
\]

\[\text{NOTE: } a(t) = \omega^2 \left(\frac{V_0}{c_0} \sin \omega t\right) = -\omega^2 y(t)
\]

\[\Rightarrow \text{acceleration is maximum when displacement } y(t) \text{ is also maximum!}
\]