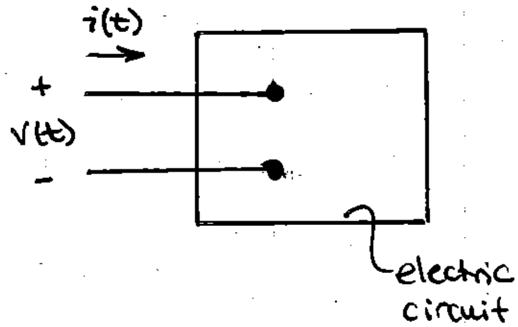
Derivatives in Electric Circuits

Electric Circuits (EE 2010)



v(t) ... voltage (volts [V])

i(t) ... current (amps [A])

voltage: $v(t) = \frac{dw}{dq}$, w ... energy (Joules [J])

q ... charge (Coulombs [C])

current: $i(t) = \frac{dq}{dt}$, t ... time (seconds [s])

power: $p(t) = \frac{dw}{dt}$, (J/s OR Watts [W])

NOTE: $p = \frac{dw}{dt} = \underbrace{\left(\frac{dw}{dq}\right)}_{v(t)}\underbrace{\left(\frac{dq}{dt}\right)}_{i(t)}$ ... chain rule

$$\boxed{\therefore\ p(t) = v(t)\, i(t)}$$

EXAMPLE: given $q(t) = \frac{1}{50}\sin(250\pi t)$

$$v(t) = 100\sin(250\pi t)$$

Find: (a) current, $i(t)$

(b) power, $p(t)$

(c) maximum power delivered

TOPS 35500

(a) current: $i(t) = \frac{dq}{dt} = \frac{d}{dt}\left(\frac{1}{50}\sin(250\pi t)\right)$

$$i(t) = \frac{1}{50}(250\pi)\cos(250\pi t)$$

$$\boxed{i(t) = 5\pi\cos(250\pi t)\ A}$$

(b) power: $p(t) = v(t)\,i(t)$

$$p(t) = (100\sin(250\pi t))(5\pi\cos(250\pi t))$$

$$p(t) = 500\pi(\sin 250\pi t)(\cos(250\pi t))$$

→ TRIG IDENTITY: $\sin\theta\cos\theta = \frac{\sin 2\theta}{2}$

$$\boxed{\therefore p(t) = 250\pi\sin(500\pi t)\ W}$$

(c) maximum power:

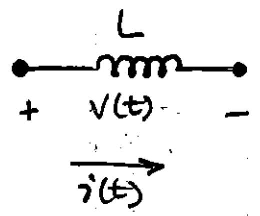
$$p(t) = 250\pi\sin(500\pi t)$$

since $-1 \le \sin(500\pi t) \le 1$,

p_{max} occurs when $\sin(500\pi t) = 1$

$$\boxed{\therefore p_{max} = 250\pi\ W}$$

EXAMPLE: Current & Voltage in an Inductor (EE 2010)



$v(t)$ ... voltage (V)

$i(t)$ ... current (A)

L ... inductance (Henrys [H])

Governing Law: $\boxed{v(t) = L\frac{di}{dt}}$

IF $L = 100\text{ mH}$ and $i(t) = t e^{-3t}$ A,

Find: (a) voltage, $v(t)$

(b) the value of the current when voltage is zero.

Solution:

(a) $v(t) = L\frac{di}{dt} = (100 \times 10^{-3})\frac{di}{dt} = 0.100\frac{di}{dt}$

$\frac{di}{dt} = \frac{d}{dt}(t e^{-3t}) \longrightarrow$ NEED PRODUCT RULE

$$\boxed{\text{PRODUCT RULE: } \frac{d}{dt}(f(t)\cdot g(t)) = f(t)\frac{dg}{dt} + g(t)\frac{df}{dt}}$$

Here, $f(t) = t$ and $g(t) = e^{-3t}$

$$\therefore \frac{di}{dt} = \frac{d}{dt}(t e^{-3t}) = t\frac{d}{dt}(e^{-3t}) + e^{-3t}\frac{d}{dt}(t)$$

$$= t(-3e^{-3t}) + e^{-3t}(1)$$

$$\frac{di}{dt} = e^{-3t}(-3t+1)$$

$$\boxed{\therefore v(t) = L\frac{di}{dt} = 0.100e^{-3t}(1-3t) \text{ volts}}$$

(b) value of <u>current</u> at $v(t) = 0$

$$v(t) = 0.100\, e^{-3t}(1-3t) = 0$$

since $e^{-3t} \neq 0$, $(1-3t) = 0$ $\quad \therefore t = \frac{1}{3}\,s$

at $t = \frac{1}{3}\,s$, $\quad i(t) = t e^{-3t}$

$$= \frac{1}{3} e^{-3(\frac{1}{3})}$$

$$\boxed{\therefore i(\tfrac{1}{3}) = \tfrac{1}{3} e^{-1} = 0.123\ A}$$

<u>NOTE</u>: above can be used to sketch $i(t)$,

at $t = \frac{1}{3}\,s$, $\quad v(t) = L\frac{di}{dt} = 0$

$\Rightarrow$ slope $\frac{di}{dt} = 0$ $\quad$ ($\therefore i = 0.123\,A$ is <u>maximum</u>!)

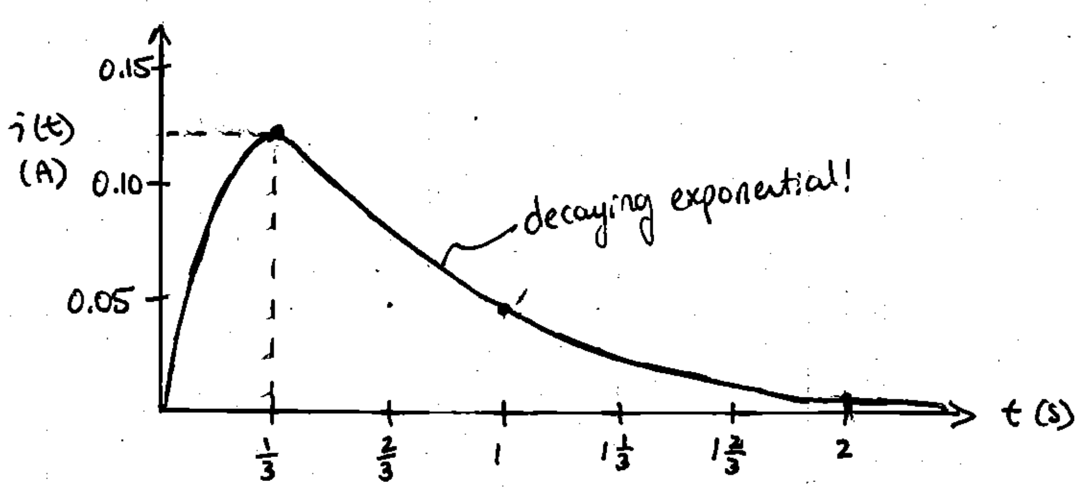
at $t = 0s$, $\quad i(0) = (0)e^{-3(0)} = 0\ A$

at $t = 1s$, $\quad i(1) = (1)e^{-3(1)} = 0.0498\ A$

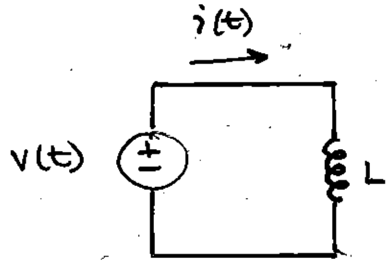
at $t = 2s$, $\quad i(2) = (2)e^{-3(2)} = 0.00496\ A$

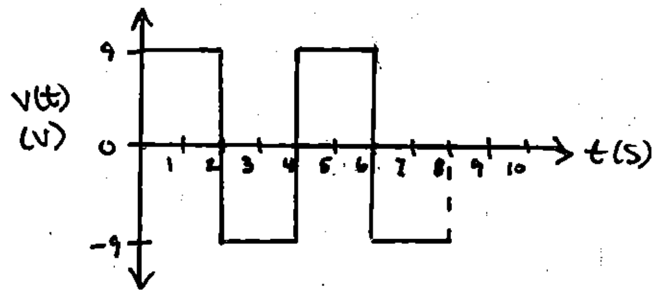
as $t \to \infty$, $\quad i(t\to\infty) = (\infty)e^{-3(\infty)} \Rightarrow i \to 0$

moves towards 0 faster!



EXAMPLE: Current & Voltage in an Inductor: (EE 2010)





For the given input voltage (square wave), sketch the current $i(t)$ and the power $p(t)$ if $L = 500\,mH$

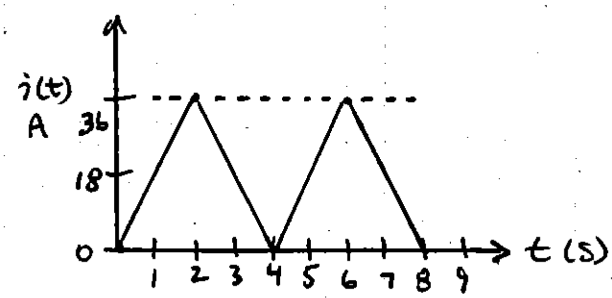
Solution: $v(t) = L\frac{di}{dt} = 0.500\frac{di}{dt}$

$$\Rightarrow \frac{di}{dt} = \frac{1}{0.500}v(t) = 2v(t)$$

(the SLOPE of $i(t)$ is twice the voltage)

since $v(t)$ is ± 9 V (constant in each interval) $i(t)$ has a slope $\frac{di}{dt}$ of ± 18 A/s

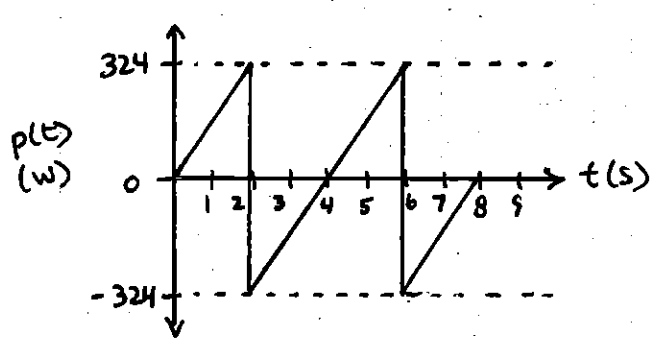
$$i_{max} = 18(2) = 36\,A$$



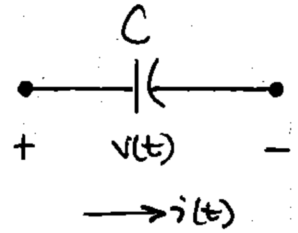
The power is $p(t) = v(t)i(t) = (\pm 9)\,i(t)$:

$$P_{max} = 9(36) = 324\,W$$

(multiply graphs of $i(t)$ & $v(t)$!)



EXAMPLE: Current & Voltage in a Capacitor (EE 2010)



$v(t)$ ... voltage (V)
$i(t)$ ... current (A)

C... capacitence (Farads [F])

Governing Law: $\boxed{i(t) = C\frac{dv}{dt}}$

If $C = 25\,\mu F$ ($\times 10^{-6}$ F), Find: the current $i(t)$ if

if $v(t) = 20e^{-500t}\sin 5000\pi t$ volts

Solution: $i(t) = C\frac{dv}{dt} = 25\times 10^{-6}\frac{dv}{dt}$

$$\frac{dv}{dt} = \frac{d}{dt}\left(20e^{-500t}\sin 5000\pi t\right)$$

Using product rule: $\frac{d}{dt}\left(f(t)\cdot g(t)\right) = f(t)\,g'(t) + g(t)\,f'(t)$

Here, $f(t) = 20e^{-500t}$ and $g(t) = \sin 5000\pi t$

$$\therefore \frac{dv}{dt} = 20e^{-500t}\frac{d}{dt}(\sin 5000\pi t) + \sin 5000\pi t\,\frac{d}{dt}(20e^{-500t})$$

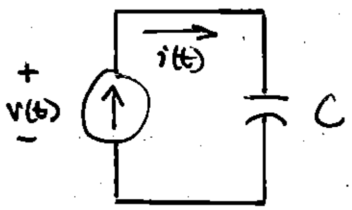
$$\frac{dv}{dt} = (20e^{-500t})(5000\pi\cos 5000\pi t) + (\sin 5000\pi t)(20(-500)e^{-500t})$$

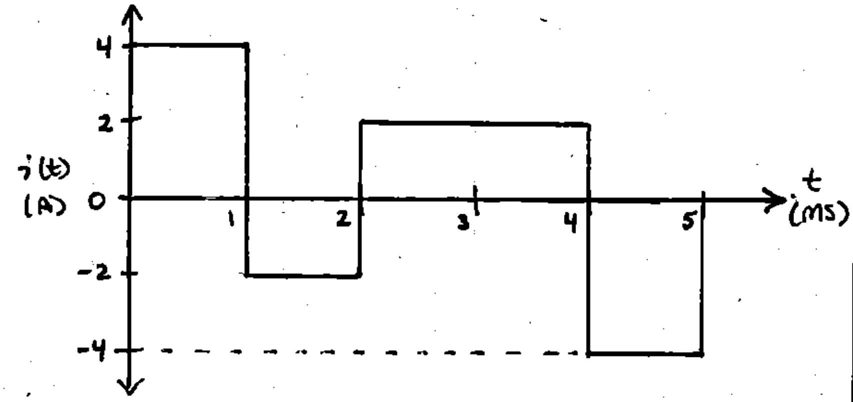
$$\frac{dv}{dt} = \underbrace{10000e^{-500t}(10\pi\cos 5000\pi t - \sin 5000\pi t)}$$

$$i(t) = C\frac{dv}{dt} = 2.5\times 10^{-6}\frac{dv}{dt} = 2.5\times 10^{-6}(\quad)$$

$$\boxed{\therefore i(t) = 0.025e^{-500t}(10\pi\cos 5000\pi t - \sin 5000\pi t)}$$

EXAMPLE: current, voltage, & charge in a capacitor (EE 2010):





Knowing that $i(t) = \frac{dq}{dt} = C\frac{dv}{dt}$, sketch the charge stored in the capacitor $q(t)$ and corresponding voltage $v(t)$ if $C = 20\,\mu F$.

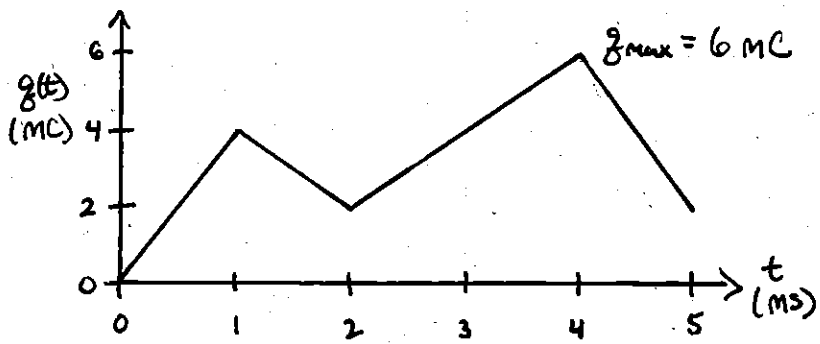
Charge: $i(t) = \frac{dq}{dt}$ (slope of $q(t)$)

$0 \le t \le 1$: $\frac{dq}{dt} = 4$

$1 \le t \le 2$: $\frac{dq}{dt} = -2$

$2 \le t \le 4$: $\frac{dq}{dt} = 2$

$4 \le t \le 5$: $\frac{dq}{dt} = -4$



voltage : $i(t) = C\frac{dv}{dt} = \frac{dq}{dt} \Rightarrow \frac{dv}{dt} = \frac{1}{C}\frac{dq}{dt}$

$\frac{dv}{dt} = \frac{1}{20\times10^{-6}}\frac{dq}{dt} = 50\times10^{3}\frac{dq}{dt}$ (same plot as above w/ y-axis scaled)