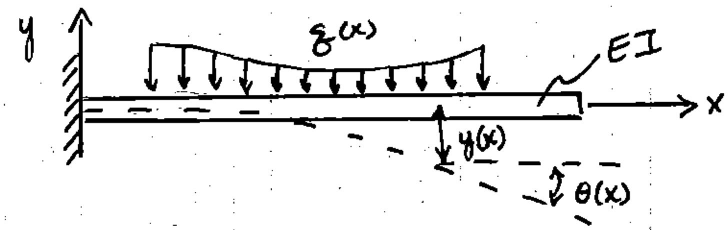
Derivatives in Strength of Materials

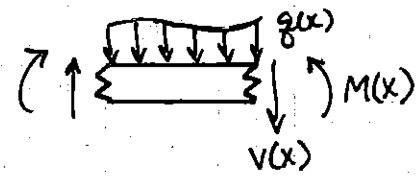
Deflection of Beams: (ME 3120; BME 3430)



$y(x)$ ... deflection (in or meters)

$\theta(x) = \frac{dy}{dx}$ ... slope (radians)

*internal forces:



$M(x) = EI\frac{d\theta}{dx} = EI\frac{d^2y}{dx^2}$ ... moment (lb·in, N·m)

$V(x) = \frac{dM}{dx} = EI\frac{d^3y}{dx^3}$ ... shear force (lb, N)

$q(x) = -\frac{dV}{dx} = -EI\frac{d^4y}{dx^4}$ ... distributed load (lb/in, N/m)

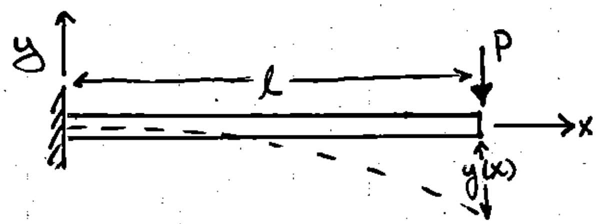
EI ... flexural rigidity ($lb \cdot in^2$, $N \cdot m^2$)

E ... elastic modulus (lb/in^2, N/m^2)

I ... second moment of area (in^4, m^4)

EXAMPLE: Deflection of Beams (ME3120 & BME3430)

Consider a cantilever beam with an end-load P



$$y(x) = \frac{P}{6EI}(x^3 - 3lx^2) \text{ m}$$

Find: the deflection & slope at the free end, $x = l$:

* deflection: $y(l) = \frac{P}{6EI}(l^3 - 3l(l)^2) = \frac{P}{6EI}(-2l^3)$

$$\Rightarrow @x = l, \quad \boxed{y = y_{MAX} = -\frac{Pl^3}{3EI} \text{ m}}$$

* slope: $\theta(x) = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{P}{6EI}(x^3 - 3lx^2)\right)$

$$\theta(x) = \frac{P}{6EI}(3x^2 - 6lx) = \frac{P}{6EI}3(x^2 - 2lx)$$

$$\theta(x) = \frac{P}{2EI}(x^2 - 2lx)$$

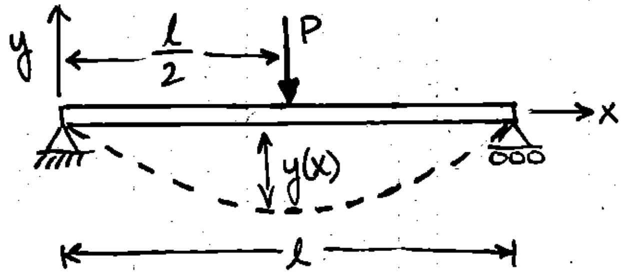
$$\Rightarrow @x = l, \quad \theta(l) = \frac{P}{2EI}(l^2 - 2l(l)) = \frac{P}{2EI}(-l^2)$$

$$\boxed{\theta(l) = \theta_{MAX} = \frac{-Pl^2}{2EI}}$$

NOTE: above values are maximums (by inspection)

EXAMPLE: Deflection of Beams (ME3120 & BME3430)

Consider a simply supported beam subjected to a central load P



$$y(x) = \frac{P}{48EI}(4x^3 - 3\ell^2 x) \text{ m}$$
(for $0 \le x \le \ell/2$)

Find: (a) the maximum deflection, y_{max}

(b) the slope at the end $x = 0$

Solution:

(a) y_{max} when $\frac{dy}{dx} = \theta(x) = 0$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{P}{48EI}(4x^3 - 3\ell^2 x)\right) = \frac{P}{48EI}(12x^2 - 3\ell^2)$$

$$\frac{dy}{dx} = \frac{3P}{48EI}(4x^2 - \ell^2) = 0$$

$$4x^2 - \ell^2 = 0 \Rightarrow x = \pm\frac{\ell}{2}$$ (negative NOT on beam)

$$\therefore y_{max} \text{ @ } x = \frac{\ell}{2}$$

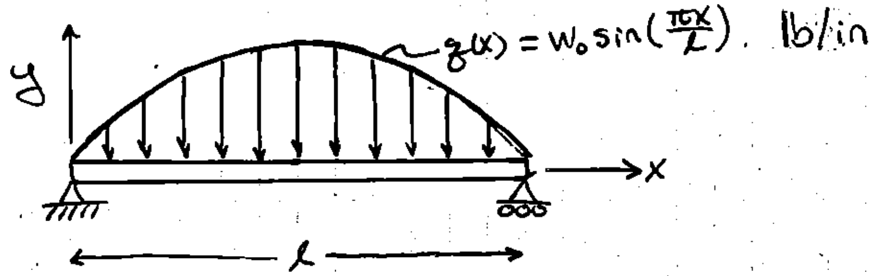
$$y_{max} = y\left(\frac{\ell}{2}\right) = \frac{P}{48EI}\left(4\left(\frac{\ell}{2}\right)^3 - 3\ell^2\left(\frac{\ell}{2}\right)\right) = \frac{P}{48EI}\left(\frac{\ell^3}{2} - \frac{3}{2}\ell^2\right)$$

$$\boxed{\therefore y_{max} = -\frac{P\ell^3}{48EI} \text{ m}}$$

(b) slope @ $x = 0$, $\theta(0) = \frac{dy}{dx}(0) = \frac{3P}{48EI}(4(0) - \ell^2) = -\frac{3P\ell^2}{48EI}$

$$\boxed{\theta(0) = \frac{P\ell^2}{16EI}}$$

EXAMPLE: Deflection of Beams (ME 3120 & BME 3430)



For the given distributed load $q(x)$, the deformation is

$$y(x) = -\frac{w_0 \ell^4}{\pi^4 EI} \sin\left(\frac{\pi x}{\ell}\right) \text{ in}$$

Find: the slope $\theta(x)$, the moment $M(x)$, and the shear force $V(x)$.

slope, $\theta(x) = \frac{dy}{dx} = -\frac{w_0 \ell^4}{\pi^4 EI}\left(\frac{\pi}{\ell}\right)\cos\left(\frac{\pi x}{\ell}\right)$

$$\boxed{\theta(x) = -\frac{w_0 \ell^3}{\pi^3 EI}\cos\left(\frac{\pi x}{\ell}\right)}$$

moment, $M(x) = EI\frac{d^2y}{dx^2} = EI\frac{d\theta}{dx} = EI\frac{d}{dx}\left(-\frac{w_0 \ell^3}{\pi^3 EI}\cos\left(\frac{\pi x}{\ell}\right)\right)$

$$\boxed{M(x) = \frac{w_0 \ell^2}{\pi^2}\sin\left(\frac{\pi x}{\ell}\right) \text{ lb}\cdot\text{in}}$$

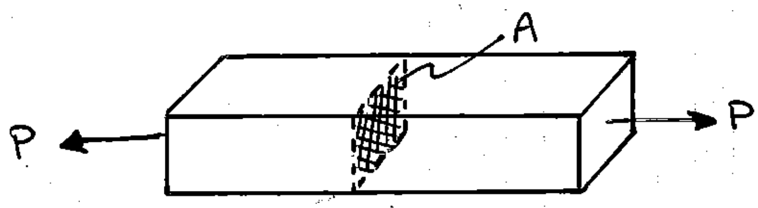
shear force, $V(x) = EI\frac{d^3y}{dx^3} = \frac{dM}{dx} = \frac{d}{dx}\left(\frac{w_0 \ell^2}{\pi^2}\sin\left(\frac{\pi x}{\ell}\right)\right)$

$$\boxed{V(x) = \frac{w_0 \ell}{\pi}\cos\left(\frac{\pi x}{\ell}\right) \text{ lb}}$$

$$q(x) = -\frac{dV}{dx} = -\frac{d}{dx}\left(\frac{w_0 \ell}{\pi}\cos\left(\frac{\pi x}{\ell}\right)\right) = w_0 \sin\left(\frac{\pi x}{\ell}\right) \checkmark$$

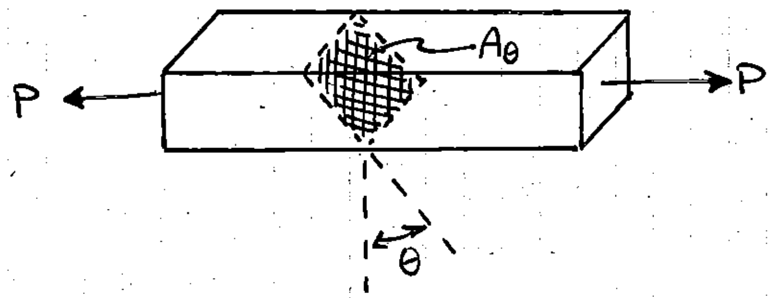
same as given in problem statement

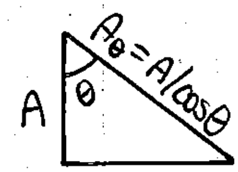
EXAMPLE: Max stress under axial loading (ME 3120 & BME 3430)



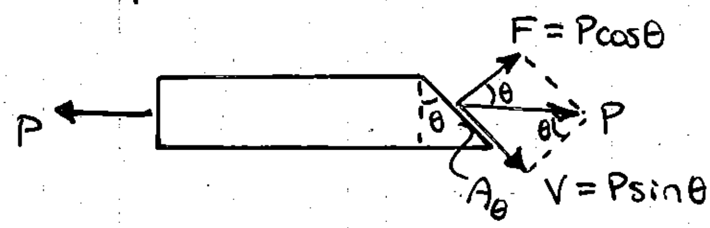
normal stress, $\sigma = \frac{P}{A}$

What stresses act on an oblique plane?

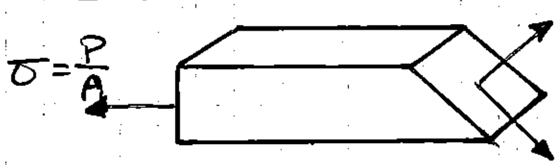




Free-Body Diagram:



Stresses:



$$\sigma_\theta = \frac{F}{A_\theta} = \frac{P\cos\theta}{A/\cos\theta} = \frac{P\cos^2\theta}{A}$$

$$\tau_\theta = \frac{V}{A_\theta} = \frac{P\sin\theta}{A/\cos\theta} = \frac{P\sin\theta\cos\theta}{A}$$

Since $\sigma = \frac{P}{A}$, $\quad \sigma_\theta = \sigma\cos^2\theta$

$\tau_\theta = \sigma\sin\theta\cos\theta$

Materials break at <u>max</u> values of σ_θ or τ_θ

For what value of θ is σ_θ <u>max</u>?

> **RECALL:** maximums & minimums of a function occur at points at which they have <u>zero slope</u>!
> * derivative = 0 & solve for pt.

Continuing, $\frac{d\sigma_\theta}{d\theta} = \sigma(2\cos\theta)(-\sin\theta) = 0 \qquad (0 \le \theta \le 90°)$

$\rightarrow$ either $\cos(\theta) = 0 \quad (\theta = 90°)$ or $\sin(\theta) = 0 \quad (\theta = 0°)$

<u>critical</u> <u>points</u>: $\theta = 90°, \theta = 0°$

* which is max? ... need $\frac{d^2\sigma_\theta}{d\theta^2}$!!!

$$\frac{d^2\sigma_\theta}{d\theta^2} = \frac{d}{d\theta}(-2\cos\theta\sin\theta) = -2\sigma[\cos\theta(\cos\theta) + \sin\theta(-\sin\theta)]$$

$$\frac{d^2\sigma_\theta}{d\theta^2} = -2\sigma(\cos^2\theta + \sin^2\theta) = -2\sigma\cos 2\theta$$

$@\theta = 0°$: $\frac{d^2\sigma_\theta}{d\theta^2} = -2\sigma < 0 \rightarrow$ <u>maximum</u> value

$@\theta = 90°$: $\frac{d^2\sigma_\theta}{d\theta^2} = -2\sigma\cos(180°) = +2\sigma > 0 \rightarrow$ <u>minimum</u> value

$$\boxed{\therefore \text{max}: @\theta = 0°, \ \sigma_{max} = \sigma\cos^2(0°) = \sigma}$$

For what value of θ is τ_θ <u>max</u>?

$$\tau_\theta = \sigma\sin\theta\cos\theta$$

$$\frac{d\tau_\theta}{d\theta} = \sigma[\sin\theta(-\sin\theta) + \cos\theta(\cos\theta)]$$

$$\frac{d\tau_\theta}{d\theta} = \sigma[\cos^2\theta - \sin^2\theta] = \sigma\cos 2\theta = 0$$

$$\cos 2\theta = 0 \Rightarrow 2\theta = 90° \Rightarrow \theta = 45°$$

checking for maximum:

$$\frac{d^2\tau_\theta}{d\theta^2} = \sigma(-\sin 2\theta)(2) = -2\sigma\sin 2\theta$$

for $0 \le \theta \le 90°, \Rightarrow 0 \le 2\theta \le 180° \Rightarrow \overbrace{\sin 2\theta > 0}^{\text{true for all } \theta \ (0 \rightarrow 180°)}$

$\therefore \frac{d^2\tau_\theta}{d\theta^2} < 0$ ... maximum

$@\theta = 45°$, $\tau_{max} = \sigma\sin 45°\cos 45° = \sigma\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \boxed{\tau_{max} = \frac{\sigma}{2} \ @ \ \theta = 45°}$