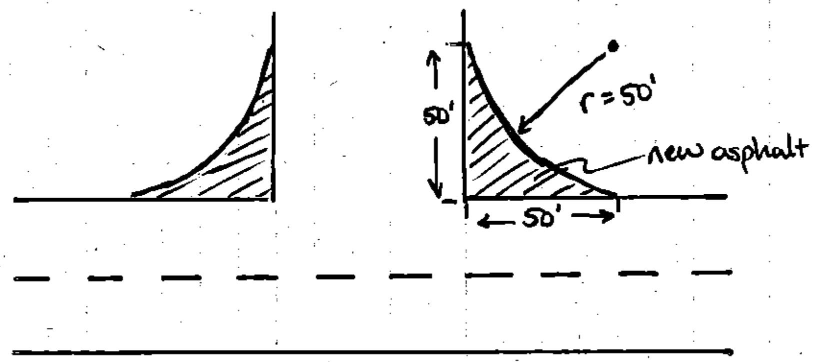
Integrals in Engineering

Integration: what is it & why do engineers need to know it?

* Based on a true story:

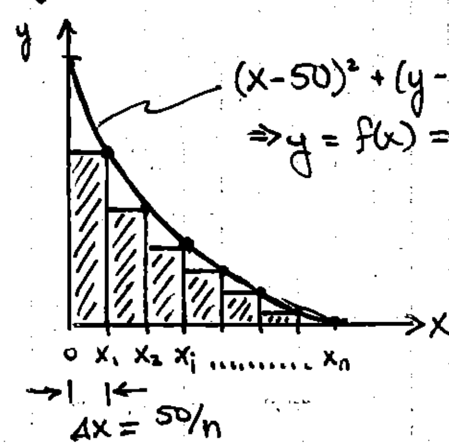
"An engineering co-op had to hire an asphalt contractor to widen the truck entrance to the DAP, Inc. headquarters in Tipp City, OH.



The asphalt company charges by the square foot, and provides an estimate based on "eyeballing" the required area of new asphalt.

The co-op asks a young engineer how he might estimate the area to make sure the quote is fair.

The young engineer proposes to estimate the area as a series of n inscribed rectangles inside the area.

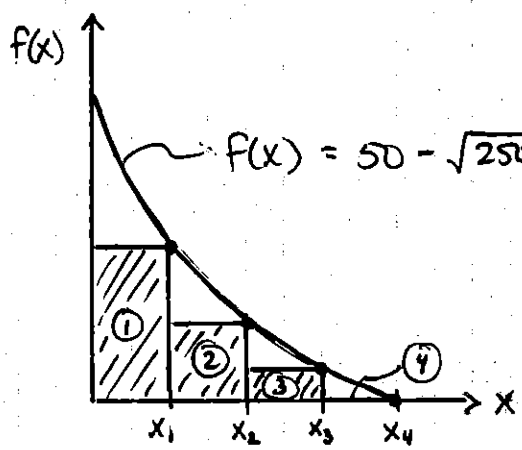


$$(x-50)^2 + (y-50)^2 = 2500$$

$$\Rightarrow y = f(x) = -\sqrt{2500 - (x-50)^2} + 50$$

$$A = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

<u>EXAMPLE</u>: $n = 4 \Rightarrow \Delta x = \frac{50}{4} = 12.5'$



$$A = \sum_{i=1}^{4} f(x_i)\Delta x$$

$$A = 12.5\left(f(x_1) + f(x_2) + f(x_3) + f(x_4)\right)$$

1. $f(x_1) = f(12.5) = 16.93$ ft.
2. $f(x_2) = f(25.0) = 6.70$ ft.
3. $f(x_3) = f(37.5) = 1.59$ ft.
4. $f(x_4) = f(50.0) = 0$ ft.

$$A = 12.5(16.93 + 6.70 + 1.59 + 0)$$

$$\boxed{\therefore A \approx 315.4 \text{ ft}^2}$$

This result <u>underestimates</u> the answer! The young engineer claims he would need an ∞ number of rectangles to get it right!

ie, $$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

In comes the old engineer, who recognizes this as the definition of the definite integral

ie, $$\underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}_{\text{area under } f(x) \text{ between } x=a \text{ and } x=b} = \underbrace{\int_a^b f(x)\,dx}_{\text{integral of } f(x) \text{ between } a \,\&\, b}$$

Also, $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ ... (1)

where $F(x)$ is the <u>antiderivative</u> of $f(x)$

Antidifferentiation:

If $F(x)$ is the antiderivative of $f(x)$, then $f(x)$ is the derivative of $F(x)$

ie, $$f(x) = \frac{d}{dx} F(x) \qquad \ldots (2)$$

EX. $f(x) = \sin(x) \longrightarrow F(x) = -\cos(x) + C$

$f(x) = x^2 \longrightarrow F(x) = \frac{1}{3}x^3 + C$

$f(x) = x^n \longrightarrow F(x) = \frac{1}{n+1} x^{n+1} + C$

NOTE: the following statements are equivalent:

$$\left.\begin{array}{l} (i)\ f(x) = \frac{d}{dx} F(x) \\ (ii)\ F(x) = \int f(x)\,dx \end{array}\right\} \text{both mean that } F(x) \text{ is the antiderivative of } f(x)$$

EX:

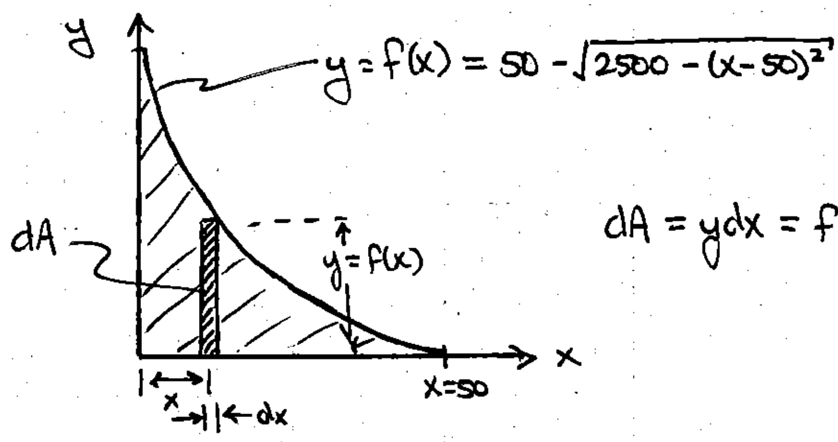
$$\left.\begin{array}{l} \int \sin(x)\,dx = -\cos(x) + C \\ \int x^2\,dx = \frac{1}{3}x^3 + C \\ \int x^n\,dx = \frac{1}{n+1} x^{n+1} + c \end{array}\right\} \text{indefinite integrals (no limits a \& b)}$$

NOTE: from (i) and (ii), differentiation and integration are inverse functions of one another

ie, $$f(x) = \frac{d}{dx} F(x) = \frac{d}{dx}\int f(x)\,dx = f(x)$$

$$F(x) = \int f(x)\,dx = \int \frac{d}{dx} F(x)\,dx = F(x)$$

→ Back to the asphalt problem:



$$dA = y\,dx = f(x)\,dx$$

The total area is the sum of all elemental areas dA,

$$A = \int dA = \int_0^{50} f(x)\,dx$$

$$A = \int_0^{50} \left(50 - \sqrt{2500 - (x-50)^2}\right) dx$$

$$A = \int_0^{50} 50\,dx - \int_0^{50} \sqrt{2500 - (x-50)^2}\,dx$$

$$A = \left[50x\right]_0^{50} - \int_0^{50} \sqrt{2500 - (x-50)^2}\,dx$$

$$A = (2500 - 0) - \underbrace{\int_0^{50} \sqrt{2500 - (x-50)^2}\,dx}_{I}$$

NEED $I = \int_0^{50} \sqrt{2500 - (x-50)^2}\,dx = 625\pi$ (MATLAB!)

Alternatively, use Table of Integrals (reference only)

$$\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + a^2/2 \sin^{-1}\left(\frac{u}{a}\right) + C$$

Let $u = x - 50 \Rightarrow du = dx$

@ $x = 0$, $u = -50$, @ $x = 50$, $u = 0$

$$I = \int_0^{50} \sqrt{2500 - (x-50)^2}\, dx = \int_{-50}^{0} \sqrt{2500 - u^2}\, dx$$

$a^2 = 2500 \therefore a = 50$

$$I = \left[\frac{u}{2}\sqrt{2500-u^2} + \frac{2500}{2}\sin^{-1}\left(\frac{u}{50}\right)\right]_{-50}^{0}$$

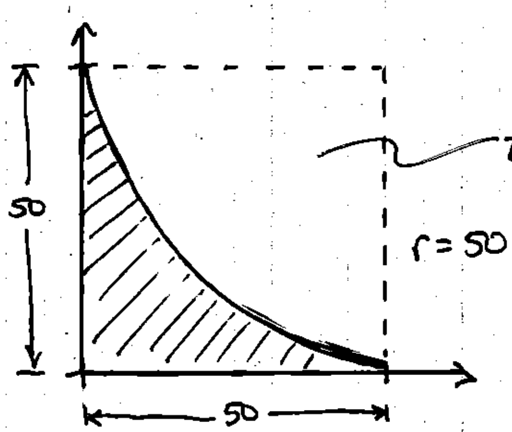
$$I = 0 + 1250\left[\underbrace{\sin^{-1}\left(\frac{0}{50}\right)}_{0} - \underbrace{\sin^{-1}\left(\frac{-50}{50}\right)}_{-\pi/2}\right) = 1250\left(\frac{\pi}{2}\right)$$

$$\therefore I = 625\pi$$

Recall $A = 2500 - I = 2500 - 625\pi = 536.5$

$$\boxed{\therefore A = 536.5 \text{ ft.}^2}$$

NOTE: the oldest engineer then arrives, who notes that this result can be determined without calculus! (b/c of simple shape!)



$\frac{1}{4}$ circle, $A = \frac{1}{4}\pi r^2$

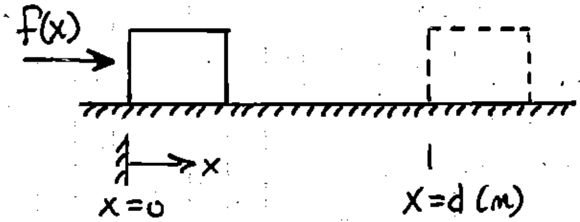
$= \frac{1}{4}\pi(50)^2$

$= 625\pi$

$\therefore A = (50)(50) - 625\pi = 2500 - 625\pi = \underline{536.5}$!

EXAMPLE: Concept of Work (PHY 2400 : ME 2120):

A variable force $f(x)$ pushes a block a distance d:



The work done on the block by the force $f(x)$ is defined as:

$$W = \int_0^d f(x)\,dx \quad (N \cdot m)$$

If $d = 1.0$ m, find the work done for

(a) $f(x) = 2x^2 + 3x + 4$ (N)

(b) $f(x) = 2\sin(\frac{\pi}{2}x) + 3\cos(\frac{\pi}{2}x)$ (N)

(c) $f(x) = 4e^{\pi x}$ (N)

Solution:

(a) $W = \int_0^d f(x)\,dx = \int_0^1 (2x^2 + 3x + 4)\,dx$

$$W = \left[2(\tfrac{1}{3})x^3 + 3(\tfrac{1}{2})x^2 + 4x\right]_0^1$$

$$W = 2(\tfrac{1}{3}) + 3(\tfrac{1}{2}) + 4 - (0 + 0 + 0)$$

$$W = 2/3 + 3/2 + 4 = 37/6 = 6.17$$

$$\boxed{\therefore W = 6.17 \ N \cdot m}$$

(b) $W = \int_0^d f(x)\,dx = \int_0^1 2\sin\left(\frac{\pi}{2}x\right) + 3\cos\left(\frac{\pi}{2}x\right) dx$

$W = 2\int_0^1 \sin\left(\frac{\pi}{2}x\right)dx + 3\int_0^1 \cos\left(\frac{\pi}{2}x\right) dx$

$W = 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\right]_0^1 + 3\left[\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)\right]_0^1$

$W = 2\left[-\frac{2}{\pi}\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} - -\frac{2}{\pi}\underbrace{\cos(0)}_{1}\right] + 3\left[\frac{2}{\pi}\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} - \frac{2}{\pi}\underbrace{\sin(0)}_{0}\right]$

$W = 2\left(0 + \frac{2}{\pi}\right) + 3\left(\frac{2}{\pi} - 0\right) = \frac{4}{\pi} + \frac{6}{\pi} = \frac{10}{\pi} = 3.18$

$$\boxed{\therefore W = 3.18 \text{ N.m}}$$

(c) $W = \int_0^d f(x)\,dx = \int_0^1 4e^{\pi x}\,dx$

$W = 4\int_0^1 e^{\pi x}\,dx = 4\left[\frac{1}{\pi}e^{\pi x}\right]_0^1$

$= 4\left[\frac{1}{\pi}e^{\pi} - \frac{1}{\pi}e^{0}\right]$

$= \frac{4}{\pi}\left(e^{\pi} - 1\right) = 28.2$

$$\boxed{\therefore W = 28.2 \text{ N.m}}$$

NOTE: In all three cases, the total distance is $d = 1.0$ m, but the work (energy) expended is completely different!