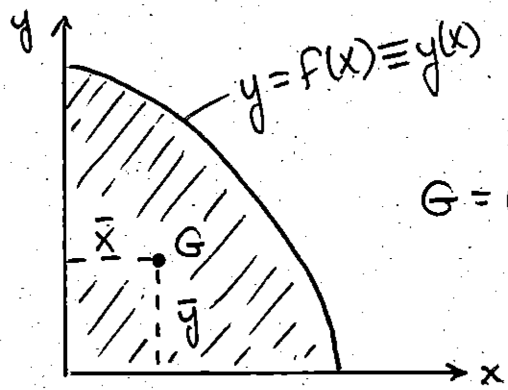
Integrals in Statics

Application of Integrals in Statics (ME2120):

Centers of Gravity (centroid):

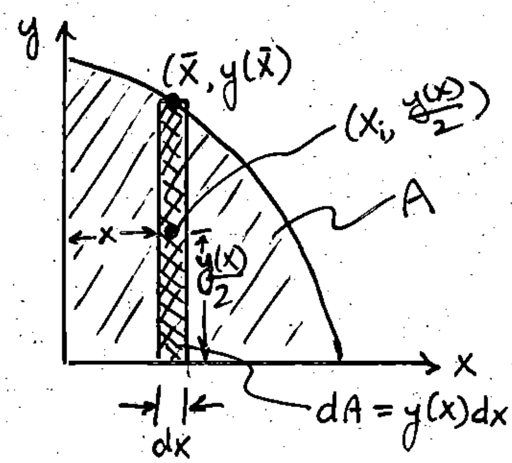


$G = (\bar{x}, \bar{y})$ ... center of gravity

$\bar{X}$ ... average x location of material $= \dfrac{\Sigma \bar{x}_i A_i}{\Sigma A_i}$

$\bar{y}$ ... average y location of material $= \dfrac{\Sigma \bar{y}_i A_i}{\Sigma A_i}$

* Consider an element of material of width dx and centroid $(x, y/2)$



$\bar{x}_i = x$

$\bar{y}_i = \dfrac{y(x)}{2}$

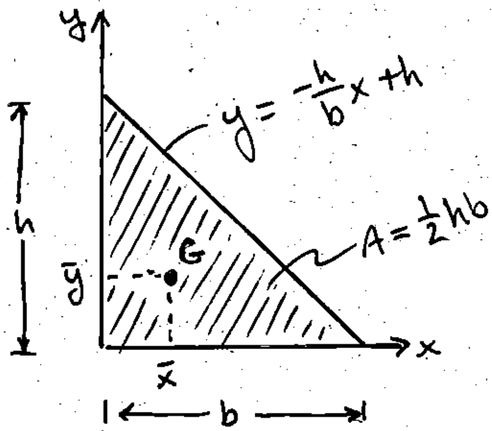
$A_i = dA = y(x)dx$

By definition,

$$\bar{x} = \frac{\Sigma \bar{x}_i A_i}{A_i} = \frac{\int x\,dA}{\int dA} = \frac{\int x\,y(x)\,dx}{\int y(x)\,dx}$$

$$\bar{y} = \frac{\Sigma \bar{y}_i A_i}{\Sigma A_i} = \frac{\int (y(x)/2)\,dA}{\int dA} = \frac{\frac{1}{2}\int (y(x))^2\,dA}{\int y(x)\,dx}$$

EXAMPLE: Centroid of Triangular Section



$$\bar{x} = \frac{\int x\,y(x)\,dx}{\underbrace{\int y(x)\,dx}_{A=\frac{1}{2}hb}} = \frac{\int_0^b x\left(-\frac{h}{b}x + h\right)dx}{\frac{1}{2}hb} = \frac{2\int_0^b \left(-\frac{h}{b}x^2 + hx\right)dx}{hb}$$

$$\bar{x} = \frac{2}{hb}\left[-\frac{h}{b}\frac{x^3}{3} + h\frac{x^2}{2}\right]_0^b$$

$$= \frac{2}{hb}\left[-h\frac{b^2}{3} + \frac{hb^2}{2}\right] = \frac{2}{hb}\left(\frac{hb^2}{6}\right) = \boxed{\frac{b}{3}}$$

$$\bar{y} = \frac{\frac{1}{2}\int (y(x))^2 dx}{\int y(x)\,dx} = \left(\frac{1}{2}\right)\frac{2}{hb}\int_0^b \left(-\frac{h}{b}x + h\right)^2 dx$$

$$\bar{y} = \frac{1}{hb}\int_0^b \left(-\frac{h}{b}x + h\right)^2 dx = \frac{1}{hb}\int_0^2 \left[\frac{h^2}{b^2}x^2 - \frac{2h}{b}xh + h^2\right]dx$$

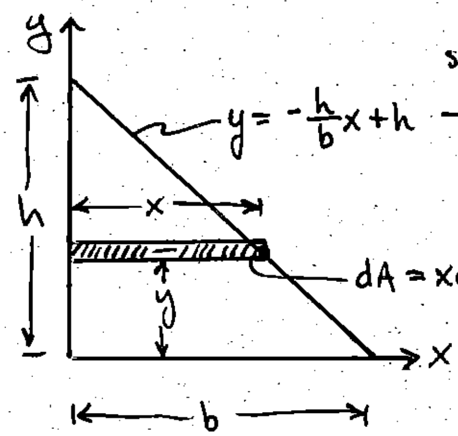
$$= \frac{1}{hb}\left[\frac{h^2}{b^2}\frac{x^3}{3} - \frac{2h^2}{b}\frac{x^2}{2} + h^2 x\right]_0^b$$

$$= \frac{1}{hb}\left[\frac{h^2 b}{3} - h^2 b + h^2 b\right] = \boxed{\frac{h}{3}}$$

$$\boxed{\therefore (\bar{x}, \bar{y}) = \left(b/3,\ h/3\right)}$$

<u>Aside</u>: we can also calculate the centroid using <u>horizontal</u> rectangles!

$$dA = g(y)\,dy$$



$y = -\frac{h}{b}x + h$ → (solve for x) → $x = -\frac{b}{h}y + b = g(y)$

$dA = x\,dy = g(y)\,dy = A_i$

$\bar{y}_i = y$

$\bar{x}_i = \frac{x}{2}$

E.g.

$$\bar{y} = \frac{\Sigma \bar{y}_i A_i}{\Sigma A_i} = \frac{\int y\,dA}{A} = \frac{\int_0^h y\left(-\frac{b}{h}y + b\right)dy}{\frac{1}{2}bh}$$

$$\bar{y} = \frac{2}{bh}\int_0^h \left(-\frac{b}{h}y^2 + by\right)dy$$

$$\bar{y} = \frac{2}{bh}\left[-\frac{b}{h}\frac{y^3}{3} + \frac{by^2}{2}\right]_0^h$$

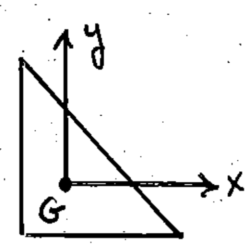
$$\bar{y} = \frac{2}{bh}\left[-\frac{bh^2}{3} + \frac{bh^2}{2}\right] = \boxed{\frac{h}{3}}$$

... same as before!

<u>Final note on centroids</u>:

$\bar{y} = \frac{\int y\,dA}{A}$ ... y coordinate of <u>centroid</u>

Suppose now that the origin of the x-y coordinate axes is located at the centroid:

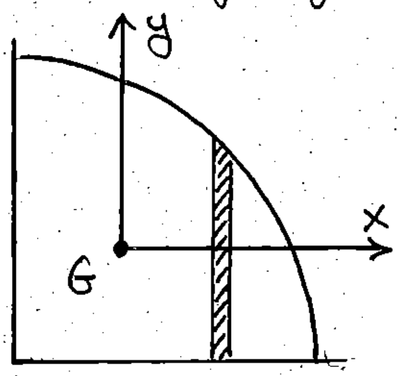


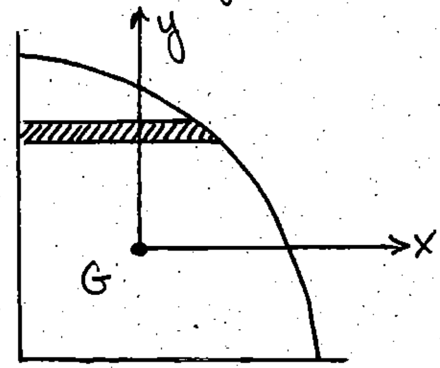
$$\bar{y} = \frac{\int y\,dA}{A} = 0$$

$$\bar{x} = \frac{\int x\,dA}{A} = 0$$

Alternative definition of centroid:

Location of the x-y origin such that $\int x\,dA = \int y\,dA = 0$!

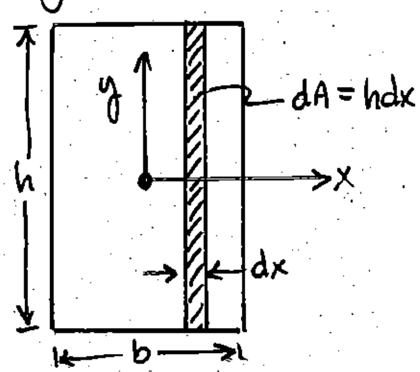




$$\underbrace{\int x\,dA}_{\text{first moment of area about y axis}} = 0 \qquad \underbrace{\int y\,dA}_{\text{first moment of area about x axis}} = 0$$

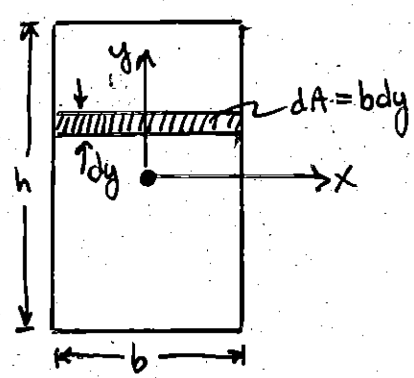
EX: Rectangular section (easy centroid)

(i)



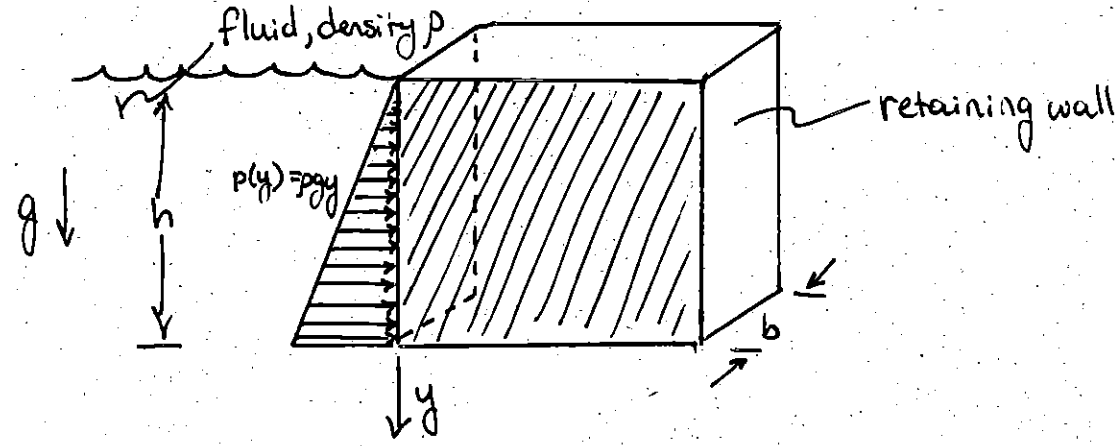
$$\int x\,dA = \int_{-b/2}^{b/2} hx\,dx = \left[\frac{hx^2}{2}\right]_{-b/2}^{b/2} = \frac{hb^2}{8} - \frac{hb^2}{8} = 0$$

(ii)



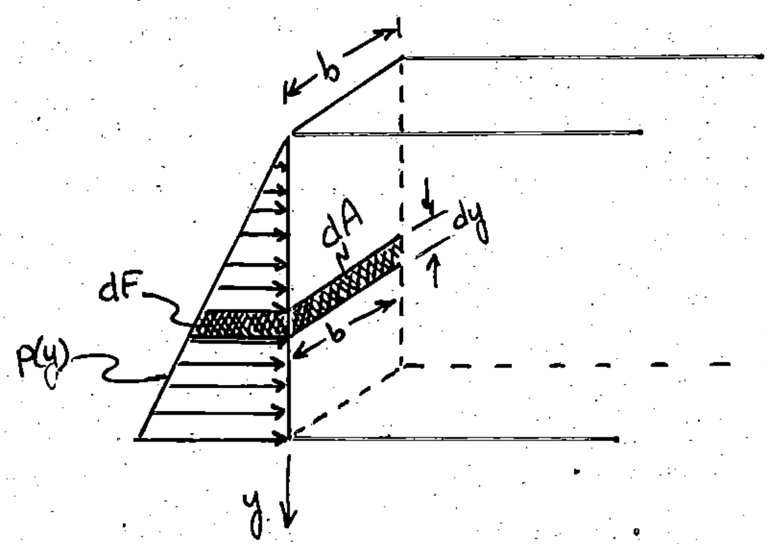
$$\int y\,dA = \int_{-h/2}^{h/2} yb\,dy = \left[\frac{by^2}{2}\right]_{-h/2}^{h/2} = \frac{bh^2}{8} - \frac{bh^2}{8} = 0$$

Distributed Loads: (e.g. hydrostatic pressure)



p … pressure (force / unit area)

What is the resultant force on the wall?



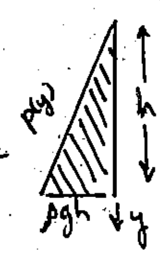
$dA = b\,dy$

$dF = p(y)\,dA = p(y)\,b\,dy$

Total Force: $F = \int dF = \int_0^h p(y)\,b\,dy = \int_0^h \rho g y\, b\,dy$

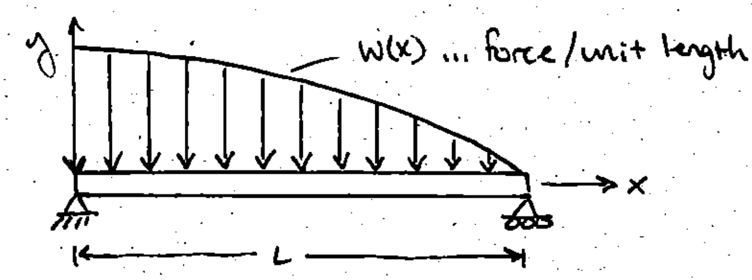
$$F = \rho g b \int_0^h y\,dy = \rho g y \left[y^2/2 \right]_0^h = \rho g b h^2/2$$

NOTE: $F = \int_0^h p(y)\,b\,dy = \underbrace{b\int_0^h p(y)\,dy}_{\text{area under } p(y) \text{ curve}} = b\left(\tfrac{1}{2}\rho g h * h\right) = \rho g b h^2/2$

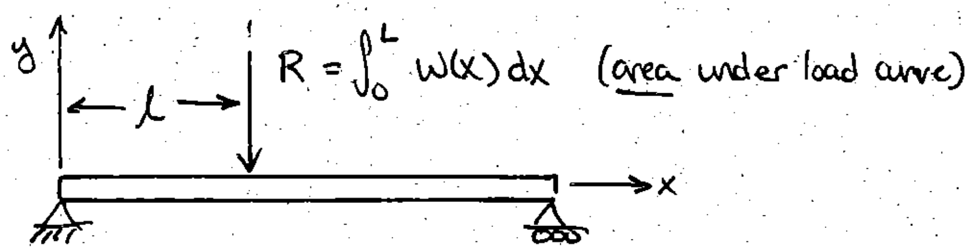


⟹ Resultant force obtained from area under the curve/load!

Distributed Loads on Beams:



Statically Equivalent Loading:



R ... resultant load

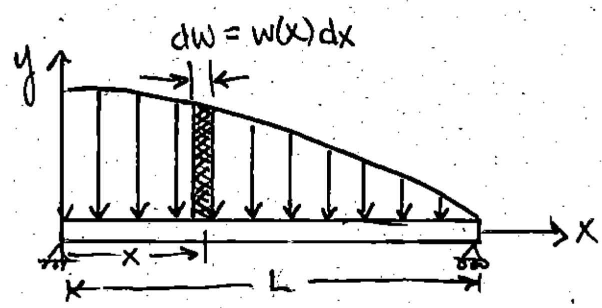
To be statically equivalent, the resultant load must have the same moment about every point as the distributed load.

e.g. moment about $x = 0$ (force times distance)

(i) Distributed Load:

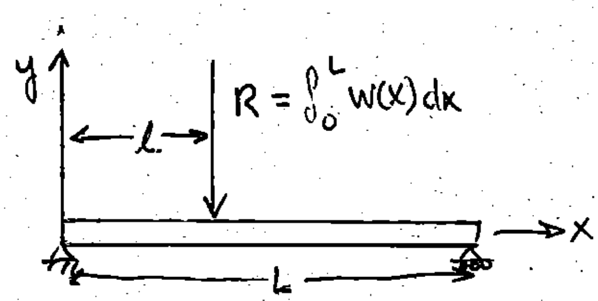
$$M_o = \int x\,dw$$

$$M_o = \int_0^L x\,w(x)\,dx \quad \ldots (1)$$



(ii) Equivalent Load:

$$M_o = R\ell = \ell \int_0^L w(x)\,dx \quad \ldots (2)$$



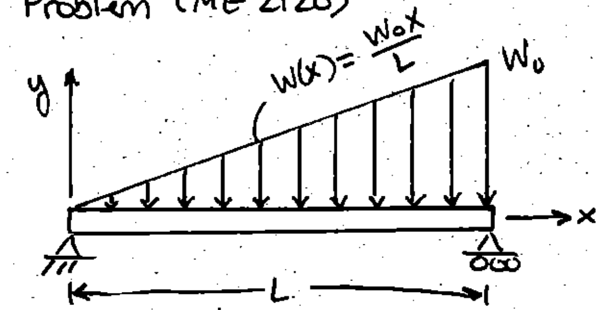
Setting (1) = (2) and solving for ℓ,

$$\Rightarrow \ell \int_0^L W(x)\,dx = \int_0^L x\,W(x)\,dx$$

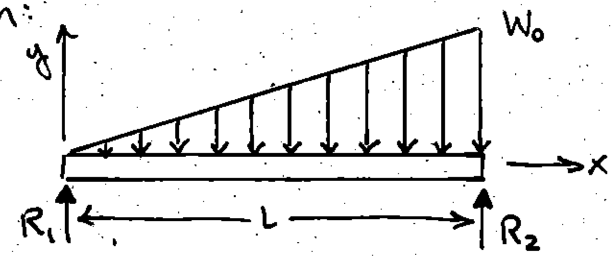
$$\Rightarrow \ell = \frac{\int_0^L x\,W(x)\,dx}{\int_0^L W(x)\,dx} \equiv \bar{X}$$ ... centroid of area under $W(x)$!

∴ we note that the resultant of a distributed load acts at the centroid!

EXAMPLE: Statics Problem (ME 2120)

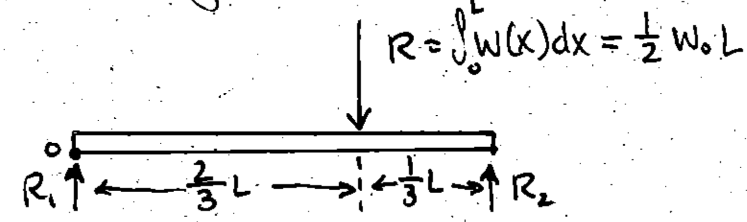


Free-Body Diagram:



Find: R_1 & R_2

Statically Equivalent Loading:



$\Sigma F_y = 0$: $R_1 + R_2 = R = \frac{1}{2} W_0 L$ ... (1)

$\Sigma M_0^{\circlearrowleft} = 0$: $R_2 L = (\frac{1}{2} W_0 L)(\frac{2L}{3}) \Rightarrow \boxed{R_2 = W_0 L/3}$

plugging into (1), $R_1 + \frac{W_0 L}{3} = \frac{W_0 L}{2} \Rightarrow R_1 = \frac{W_0 L}{2} - \frac{W_0 L}{3} = \boxed{R_1 = \frac{W_0 L}{6}}$