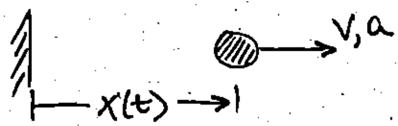
Integrals in Dynamics

Application of Integrals in Dynamics (PHY 2400 ; ME2210 ; BME3212)



$x(t)$ ... position (m)

$v(t) = \frac{dx}{dt}$ ... velocity (m/s)

$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ ... acceleration (m/s^2)

Given $a(t)$, both $v(t)$ and $x(t)$ can be determined by integration:

$$\frac{dv}{dt} = a(t)$$

Integrating both sides between $t=0$ and any time t:

$$\int_0^t \frac{dv}{dt}\,dt = \int_0^t a(t)\,dt$$

by definition, $\int \frac{dv}{dt}\,dt = v(t)$!

$$\Rightarrow \quad [v(t)]_0^t = \int_0^t a(t)\,dt$$

$$v(t) - v(0) = \int_0^t a(t)\,dt$$

$$\Rightarrow \quad \boxed{v(t) = \int_0^t a(t)\,dt + v(0)}$$

$v(0)$ ... initial velocity

Given $v(t)$, $x(t)$ can be determined by integration:

$$\frac{dx}{dt} = v(t)$$

Integrating both sides between $t=0$; any time t,

$$\int_0^t \frac{dx}{dt}\,dt = \int_0^t v(t)\,dt$$

$$\int_0^t \frac{dx}{dt}\,dt = \int_0^t v(t)\,dt$$

by definition, $\int \frac{dx}{dt}\,dt = x(t)$!

$$\Rightarrow \quad [x(t)]_0^t = \int_0^t v(t)\,dt$$

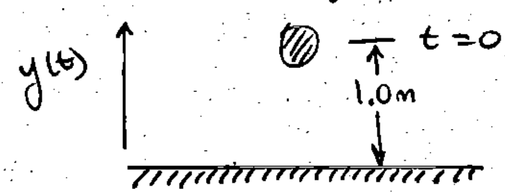
$$x(t) - x(0) = \int_0^t v(t)\,dt$$

$$\boxed{x(t) = \int_0^t v(t)\,dt + x(0)}$$

$x(0)$ ... <u>initial position</u>

<u>EXAMPLE</u>:

A ball is dropped from a height of 1.0 m at $t=0$ seconds:



$\downarrow$ $a = g = -9.81\ m/s^2$

<u>Find</u>: $v(t)$, $y(t)$, and time to impact:

<u>Solution</u>: $\frac{dv}{dt} = a(t) = -9.81$

Integrating both sides,

$$\int_0^t \frac{dv}{dt}\,dt = \int_0^t -9.81\,dt$$

$$[v(t)]_0^t = [-9.81t]_0^t$$

$$v(t) - v(0) = -9.81t - 0$$

$$v(t) = -9.81t + v(0)$$

* Here $v(0) = 0$ (dropped from rest)

$$\boxed{\therefore v(t) = -9.81t \ \frac{m}{s}}$$

Given $v(t)$, we can integrate again to get $y(t)$:

$$\frac{dy}{dt} = v(t)$$

Integrating both sides,

$$\int_0^t \frac{dy}{dt}\,dt = \int_0^t v(t)\,dt$$

$$\Big[y(t)\Big]_0^t = \int_0^t -9.81t\,dt$$

$$y(t) - y(0) = \left[-\frac{9.81}{2}t^2\right]_0^t$$

$$y(t) = -4.905t^2 - y(0) = y(0) - 4.905t^2$$

Here, $y(0) = 1.0$ m (initial height)

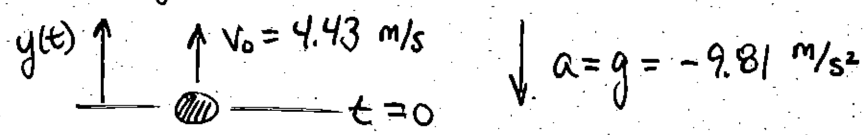
$$\boxed{y(t) = 1.0 - 4.905t^2 \text{ m}}$$

<u>Impact</u>: when $y(t) = 1.0 - 4.905t^2 = 0$

$$4.905t^2 = 1.0$$

$$t = \sqrt{\frac{1.0}{4.905}} = \underline{0.452\text{ s}}$$

<u>EXAMPLE</u>: Suppose the ball is thrown upwards with an initial velocity $v(0) = V_0 = 4.43$ m/s



<u>Find</u>: $v(t)$ and $y(t)$

<u>Solution</u>: $a = \frac{dv}{dt} = -9.81$

Integrating, $$\int_0^t \frac{dv}{dt}\,dt = \int_0^t -9.81\,dt$$

$$[v(t)]_0^t = [-9.81t]_0^t$$

$$v(t) - v(0) = -9.81t - 0$$

Here, $v(0) = 4.43$ m/s

$$\Rightarrow \boxed{v(t) = 4.43 - 9.81t \text{ m/s}}$$

Given $v(t)$, integrate to get $y(t)$:

$$\frac{dy}{dt} = v(t) = 4.43 - 9.81t$$

Integrating, $$\int_0^t \frac{dy}{dt}\,dt = \int_0^t 4.43 - 9.81t\,dt$$

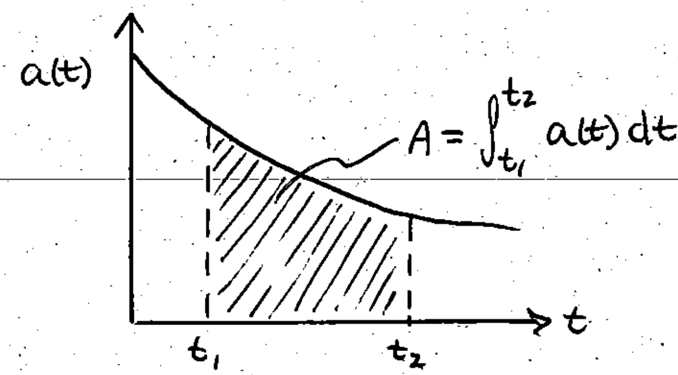
$$[y(t)]_0^t = [4.43t - \tfrac{9.81}{2}t^2]_0^t$$

$$y(t) - y(0) = 4.43t - 4.905t$$

Here $y(0) = 0$, $\boxed{y(t) = 4.43t - 4.905t^2 \text{ m}}$

Graphical Interpretation:

The velocity $v(t)$ can be determined as the area under the graph of $a(t)$:



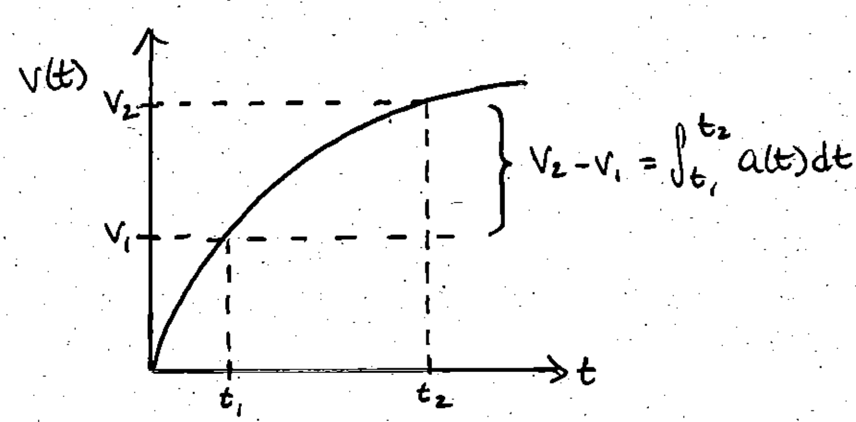
By definition, $a(t) = \frac{dv}{dt}$

Integrating both sides between t_1 and t_2,

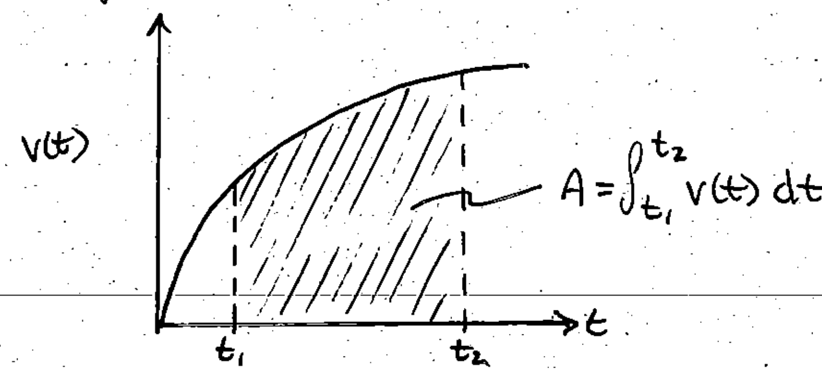
$$\int_{t_1}^{t_2} a(t)\,dt = \int_{t_1}^{t_2} \frac{dv}{dt}\,dt$$

$$\int_{t_1}^{t_2} a(t)\,dt = \left[v(t)\right]_{t_1}^{t_2}$$

$$\underbrace{\int_{t_1}^{t_2} a(t)\,dt}_{\text{area under } a(t) \text{ between } t_1 \,\&\, t_2} = V(t_2) - V(t_1) = \underbrace{V_2 - V_1}_{\text{change in } v(t) \text{ between } t_1 \,\&\, t_2}$$



Similarly, the position $x(t)$ can be determined as the area under the velocity curve $v(t)$:



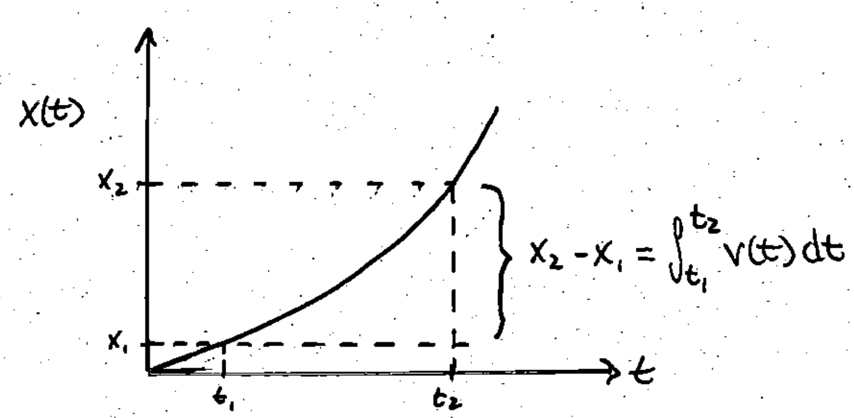
By definition, $v(t) = \frac{dx}{dt}$

Integrating both sides between t_1 and t_2,

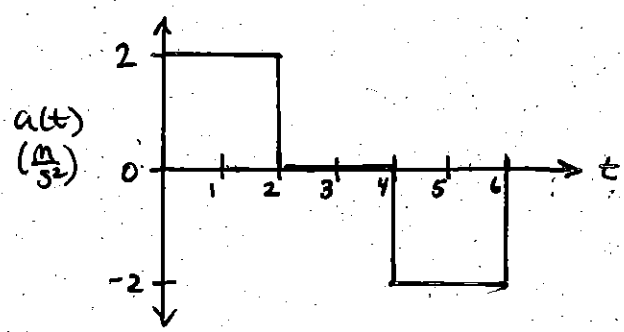
$$\int_{t_1}^{t_2} v(t)\,dt = \int_{t_1}^{t_2} \frac{dx}{dt}\,dt$$

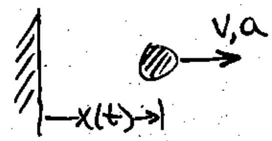
$$\int_{t_1}^{t_2} v(t)\,dt = \left[x(t)\right]_{t_1}^{t_2}$$

$$\underbrace{\int_{t_1}^{t_2} v(t)\,dt}_{\text{area under } v(t) \text{ between } t_1 \,\&\, t_2} = x(t_2) - x(t_1) = \underbrace{x_2 - x_1}_{\text{change in } x(t) \text{ between } t_1 \,\&\, t_2}$$



<u>EXAMPLE</u>: the acceleration of a particle is measured as:





Knowing the particle starts from rest, sketch the velocity $v(t)$ and the position $x(t)$ <u>using integrals</u>.

<u>Solution</u>: begin with $v(t)$, knowing $v(0) = 0$ and $v(t) = \int a(t)\,dt$

<u>$0 \le t \le 2s$</u>: $a(t) = 2$ (constant)

$$\Rightarrow v(t) = \int a(t)\,dt \quad \text{(straight line)}$$

also,

$$V_2 - V_0 = \underbrace{\int_0^2 a(t)\,dt}_{\text{area under } a(t)} = (2)(2) = 4$$

$$V_2 = V_0 + 4 = 0 + 4 = \underline{\underline{4 \tfrac{m}{s}}}$$

<u>$2 \le t \le 4s$</u>: $a(t) = 0 \Rightarrow v(t) = \int a(t)\,dt$ (constant)

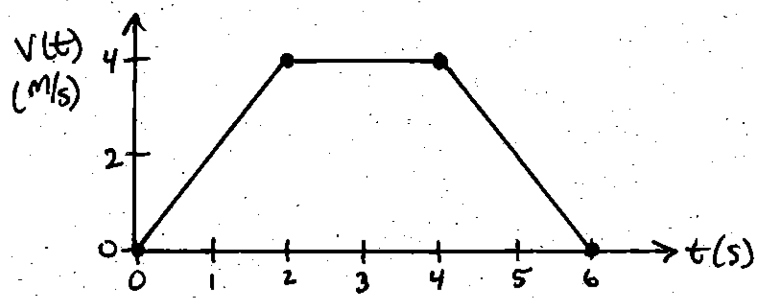
$$V_4 - V_2 = \underbrace{\int_2^4 a(t)\,dt}_{0} \Rightarrow V_4 = V_2 = \underline{\underline{4 \ m/s}}$$

<u>$4 \le t \le 6s$</u>: $a(t) = -2$ (constant)

$$v(t) = \int a(t)\,dt \quad \text{(straightline)}$$

$$V_6 - V_4 = \underbrace{\int_4^6 a(t)\,dt}_{\text{area under } a(t)} = (-2)(2) = -4$$

$$V_6 - V_4 = -4 \Rightarrow V_6 = V_4 - 4 = 4 - 4 = \underline{\underline{0 \ m/s}}$$



Now we use v(t) to sketch x(t) knowing $x(t) = \int v(t)\,dt$

<u>$0 \le t \le 2s$</u>: v(t) is <u>linear</u>, $x(t) = \int v(t)\,dt$ (quadratic)

also, $X_2 - X_0 = \underbrace{\int_0^2 v(t)\,dt}_{\text{area under } v(t)} = \frac{1}{2}(2)(4) = 4$

$$X_2 = X_0 + 4 = 0 + 4 = \underline{\underline{4}}\ m$$

<u>$2 \le t \le 4s$</u>: v(t) (constant), $x(t) = \int v(t)\,dt$ (linear)

also, $X_4 - X_2 = \int_2^4 v(t)\,dt = (2)(4) = 8$

$$X_4 = X_2 + 8 = 8 + 4 = \underline{\underline{12\,m}}$$

<u>$4 \le t \le 6s$</u>: v(t) (linear), $x(t) = \int v(t)\,dt$ (quadratic)

also, $X_6 - X_4 = \int_4^6 v(t)\,dt = \frac{1}{2}(2)(4) = 4$

$$X_6 = 4 + X_4 = 4 + 12 = \underline{\underline{16}}\ m$$