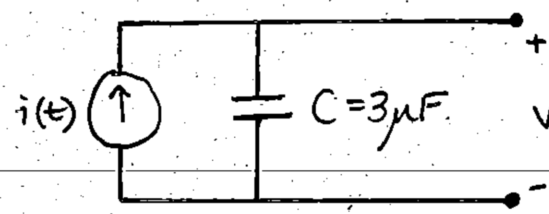
Integrals in Electric Circuits

Application of Integrals in Electric Circuits: (EE 2010)

EXAMPLE: current, voltage, and stored energy in a capacitor



$$i(t) = C\frac{dv}{dt}$$

at $t = 0$, $v(0) = 0$ (zero initial voltage)

For $t \geq 0$, $i(t) = 24e^{-40t}$ mA

Find: (a) the voltage $v(t)$

(b) stored energy, $W(t)$, and show $W(t) = \frac{1}{2}Cv^2(t)$

Solution:

(a) $i(t) = C\frac{dV}{dt} \Rightarrow \frac{dV}{dt} = \frac{1}{C}i(t)$

Integrating both sides,

$$\int_0^t \frac{dv}{dt}dt = \int_0^t \frac{1}{C}i(t)dt$$

$$[v(t)]_0^t = \frac{1}{C}\int_0^t i(t)dt$$

$$v(t) - v(0) = \frac{1}{C}\int_0^t i(t)dt$$

$$\boxed{v(t) = v(0) + \frac{1}{C}\int_0^t i(t)dt}$$

Here, $v(0) = 0$ $\quad C = 3\times10^{-6}\,F$

$$i(t) = 24e^{-40t}\,mA = 0.024e^{-40t}\,A$$

$$\Rightarrow v(t) = 0 + \frac{1}{3\times10^{-6}}\int_0^t 0.024e^{-40t}\,dt$$

$$v(t) = \frac{0.024}{3\times10^{-6}}\int_0^t e^{-40t}\,dt$$

$$v(t) = 8000\left[-\frac{1}{40}e^{-40t}\right]_0^t$$

$$v(t) = -\frac{8000}{40}\left(e^{-40t}-1\right)$$

$$v(t) = -200\left(e^{-40t}-1\right)$$

$$\boxed{\therefore v(t) = 200 - 200e^{-40t}\ \text{volts}}$$

(b) Recall $p(t) = \frac{dW}{dt}$ (energy per unit time)

Integrating both sides,

$$\int_0^t \frac{dW}{dt}\,dt = \int_0^t p(t)\,dt$$

$$\left[W(t)\right]_0^t = \int_0^t p(t)\,dt$$

$$W(t) - W(0) = \int_0^t p(t)\,dt$$

$$\boxed{W(t) = W(0) + \int_0^t p(t)\,dt}$$

Here, $W(0) = 0$ (no current for $t>0$)

Also, $p(t) = v(t)\,i(t)$

$$\therefore p(t) = \left(200 - 200e^{-40t}\right)\left(0.024e^{-40t}\right)$$

$$p(t) = (200 - 200e^{-40t})(0.024e^{-40t})$$

$$p(t) = (200)(0.024e^{-40t}) - (200e^{-40t})(0.024e^{-40t})$$

$$p(t) = 4.8e^{-40t} - 4.8e^{-80t}$$

Thus, $w(t) = 0 + \int_0^t (4.8e^{-40t} - 4.8e^{-80t})\,dt$

$$w(t) = 4.8\left[-\tfrac{1}{40}e^{-40t}\right]_0^t - 4.8\left[-\tfrac{1}{80}e^{-80t}\right]_0^t$$

$$w(t) = -\frac{4.8}{40}(e^{-40t} - 1) - \frac{-4.8}{80}(e^{-80t} - 1)$$

$$w(t) = -0.120e^{-40t} + 0.120 + 0.06e^{-80t} - 0.06$$

$$\boxed{w(t) = 0.06e^{-80t} - 0.120e^{-40t} + 0.06 \text{ J}}$$

NOTE: $\frac{1}{2}CV^2(t) = \frac{1}{2}(3\times10^{-6})(200 - 200e^{-40t})^2$

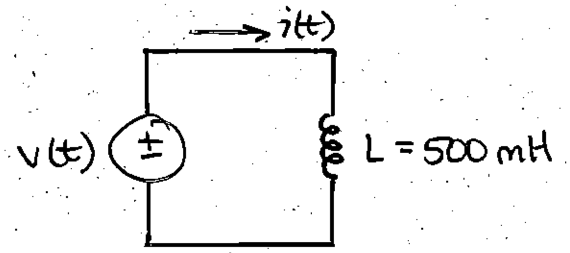
$$= 1.5\times10^{-6}\left((200)^2 - 2(200)(200)e^{-40t} + (200e^{-40t})^2\right)$$

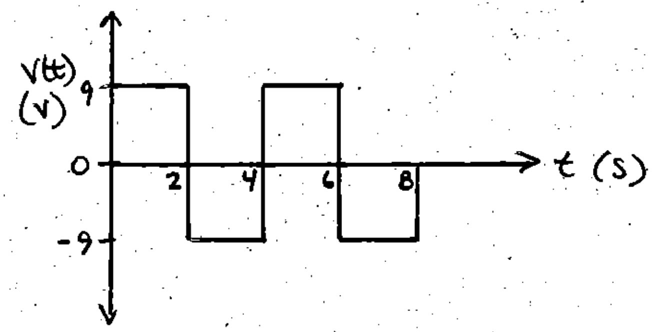
$$= 1.5\times10^{-6}(40000 - 80000e^{-40t} + 40000e^{-80t})$$

$$= 0.06 - 0.120e^{-40t} + 0.06e^{-80t} = w(x)$$

Thus it is proven, $w(t) = \frac{1}{2}CV^2(t)$!

<u>EXAMPLE</u>: current and voltage in an inductor





Knowing $v(t) = L\frac{di}{dt}$, plot the current $i(t)$ using <u>integrals</u>.

<u>Solution</u>: $\frac{di}{dt} = \frac{1}{L} v(t)$

Integrating both sides between 0 and t,

$$\int_0^t \frac{di}{dt}\,dt = \frac{1}{L}\int_0^t v(t)\,dt$$

$$[i(t)]_0^t = \frac{1}{L}\int_0^t v(t)\,dt$$

$$\boxed{i(t) - i(0) = \frac{1}{L}\int_0^t v(t)\,dt}$$

also, between any two times t_1 and t_2,

$$\underbrace{i_2 - i_1}_{\text{change in } i(t) \text{ between } t_1 \,\&\, t_2} = \frac{1}{L}\underbrace{\int_{t_1}^{t_2} v(t)\,dt}_{\text{area under } v(t) \text{ from } t_1 \text{ to } t_2}$$

$0 \le t \le 2\,s$: $v(t) = 9$ volts ... constant

$\Rightarrow i(t) = \frac{1}{L}\int v(t)dt$ ... straight line

also, $i(2) - i(0) = \frac{1}{L}\int_0^2 v(t)dt$

$= \frac{1}{0.50}(2)(9) = 36$

$i(2) = i(0) + 36 = 0 + 36 = 36\,A$

$2 \le t \le 4\,s$: $v(t) = -9$ volts ... constant

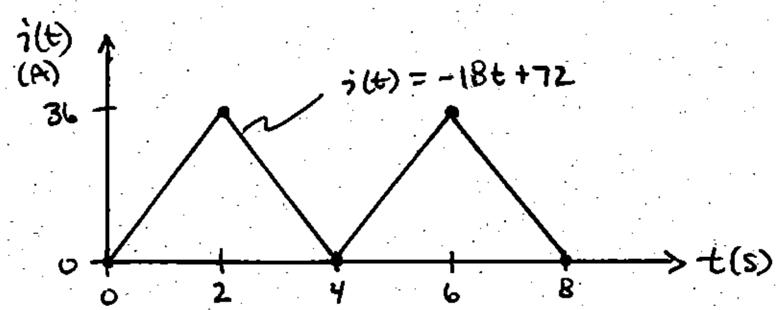
$\Rightarrow i(t) = \frac{1}{L}\int v(t)\,dt$ ... straight line

also, $i(4) - i(2) = \frac{1}{L}\int_2^4 v(t)\,dt$

$= \frac{1}{0.5}(2)(-9) = -36$

$i(4) = i(2) - 36 = 36 - 36 = 0$

Results are identical for $4 \le t \le 6s$ and $6 \le t \le 8s$



NOTE: The equation of the line in each interval can also be obtained by integration.

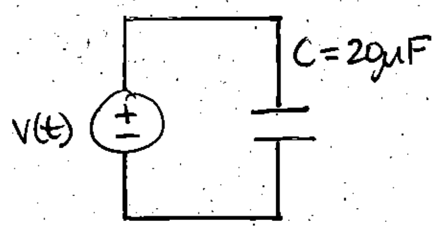
e.g. $2 \le t \le 4s$: $\int_2^t \frac{di}{dt}dt = \frac{1}{L}\int_2^t v(t)dt$

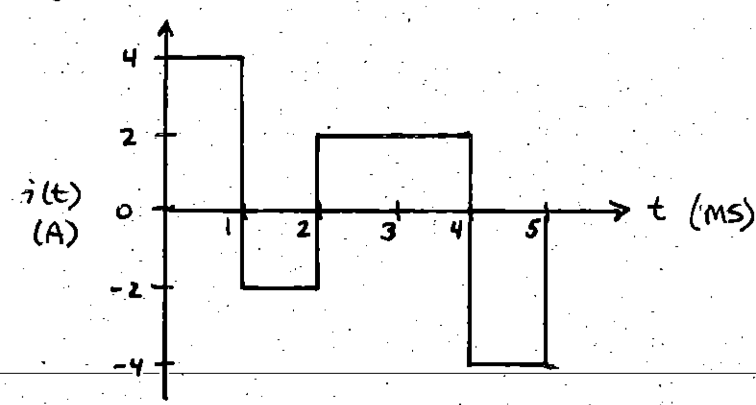
$[i(t)]_2^t = i(t) - i(2) = \frac{1}{0.5}\int_2^t -9\,dt$

$i(t) = i(2) + 2[-9t]_2^t$

$\Rightarrow i(t) = i(2) + 2(-9t - -9(2)) = 36 - 18t + 36 = \boxed{i(t) = -18 + 72\ A}$

<u>EXAMPLE</u>: Current and Voltage in a Capacitor





Sketch voltage $v(t)$, knowing $i(t) = C\frac{dv}{dt}$ or $v(t) = \frac{1}{C}\int i(t)\,dt$

<u>$0 \le t \le 1$ ms</u>: $i(t) = 4$ ... constant $\therefore v(t) = \frac{1}{C}\int i(t)\,dt$ ... straight line

also, $v(1) - v(0) = \frac{1}{C}\int_0^1 i(t)\,dt = \frac{1}{20\times10^{-6}}(1\times10^{-3})(4) = 200\text{V}$

$v(1) = v(0) + 200 = 0 + 200 = \underline{200 \text{ volts}}$

<u>$1 \le t \le 2$ ms</u>: $i(t) = -2$ ... constant $\therefore v(t) = \frac{1}{C}\int i(t)\,dt$ ... straight line

also, $v(2) - v(1) = \frac{1}{C}\int_1^2 i(t)\,dt = \frac{1}{20\times10^{-6}}(1\times10^{-3})(-2) = -100$

$v(2) = v(1) + (-100) = 200 - 100 = \underline{100 \text{ volts}}$

<u>$2 \le t \le 4$ ms</u>: $i(t) = 2$ ... constant $\therefore v(t) = \frac{1}{C}\int i(t)\,dt$ ... straight line

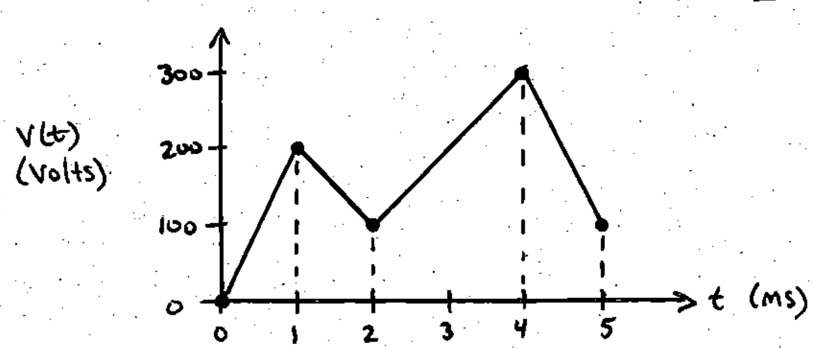
also, $v(4) - v(2) = \frac{1}{C}\int_2^4 i(t)\,dt = \frac{1}{20\times10^{-6}}(1\times10^{-3})(2) = 200$

$v(4) = v(2) + 200 = 100 + 200 = \underline{300 \text{ volts}}$

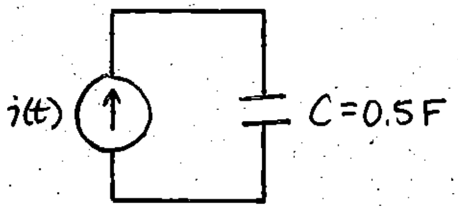
<u>$4 \le t \le 5$ ms</u>: $i(t) = -4$ ... constant $\therefore v(t) = \frac{1}{C}\int i(t)\,dt$ ... straight line

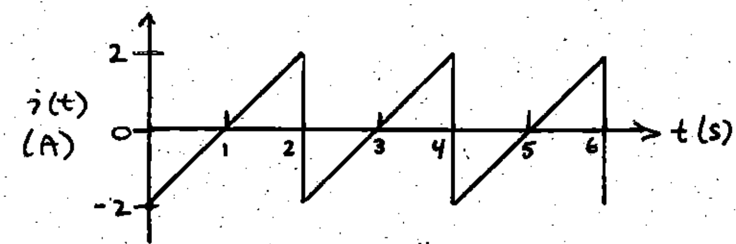
also, $v(5) - v(4) = \frac{1}{C}\int_4^5 i(t)\,dt = \frac{1}{20\times10^{-6}}(1\times10^{-3})(-4) = -200$

$v(5) = v(4) + 200 = 300 - 200 = \underline{100 \text{ volts}}$



EXAMPLE: current and voltage in a capacitor





"Sawtooth" current

Knowing $i(t) = C\frac{dV}{dt}$ $\left(V(t) = \frac{1}{C}\int i(t)dt\right)$, plot the voltage $V(t)$.
Assume $V_0 = V(0) = 0$

$0 \le t \le 1s$: $i(t)$ ... linear $\therefore V(t) = \frac{1}{C}\int i(t)dt$ ... quadratic

also, $V(1) - V(0) = \frac{1}{C}\int_0^1 i(t)dt = \frac{1}{0.5}(\frac{1}{2})(1)(-2) = -2$

$V(1) = V(0) - 2 = 0 - 2 = -2$ volts

also, since $\frac{dV}{dt} = \frac{1}{C}i(t)$ ... slope of $V(t)$

$@t=0, \frac{dV}{dt} = \frac{1}{0.5}(-2) = -4$
$@t=1, \frac{dV}{dt} = \frac{1}{0.5}(0) = 0$
} $V(t)$ is decreasing w/ slope of zero @ $t = 1s$.

$1 \le t \le 2s$: $i(t)$ ... linear $\therefore V(t) = \frac{1}{C}\int i(t)dt$ ... quadratic

also, $V(2) - V(1) = \frac{1}{C}\int_1^2 i(t)\,dt = \frac{1}{0.5}(\frac{1}{2})(1)(2) = 2$

$V(2) = V(1) + 2 = -2 + 2 = 0$ volts

also since $\frac{dV}{dt} = \frac{1}{C}i(t)$ ... slope of $V(t)$

$@t=1, \frac{dV}{dt} = \frac{1}{0.5}(0) = 0$
$@t=2, \frac{dV}{dt} = \frac{1}{0.5}(2) = 4$
} $V(t)$ is increasing w/ zero slope @ $t = 1s$.

Results are identical for remaining intervals (periodic function!)