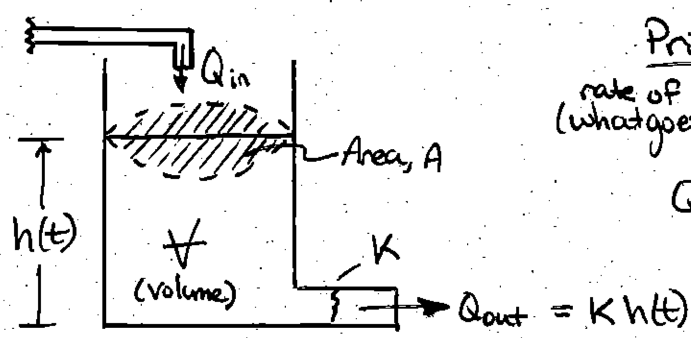
Differential Equations in Engineering

The Leaking Bucket Problem:



Principle:

$$\left(\begin{array}{c}\text{rate of}\\ \text{what goes in}\end{array}\right) - \left(\begin{array}{c}\text{rate of}\\ \text{what comes out}\end{array}\right) = \begin{array}{c}\text{rate of}\\ \text{change in}\\ \text{volume}\end{array}$$

$$Q_{in} - Q_{out} = \frac{d\forall}{dt}$$

From governing principle: $\frac{d\forall}{dt} + Q_{out} = Q_{in}$, $\frac{d\forall}{dt} = A\frac{dh}{dt}$

$$\boxed{\Rightarrow A\frac{dh}{dt} + K\,h(t) = Q_{in}}$$ ← 1^{st} order diff. eqn!

NOTE 1: Sometimes it is convenient to denote the derivative with respect to time (ie, dh/dt) with a dot ($\cdot$) over the function. So the differential equation above may be written as:

$$A\dot{h}(t) + K\,h(t) = Q_{in}$$

NOTE 2: The solution to a differential equation is a function that can be substituted along with its proper derivatives into the original differential equation to make the statement true! (LHS = RHS) This solution is generally made up of two parts.

(1) Transient (aka. complementary or homogeneous)

(2) Steady-state (aka. particular)

$A\frac{dh}{dt} + Kh(t) = Q_{in}$ also, $A\dot{h}(t) + Kh(t) = Q_{in}$

(1) Transient (complementary / homogeneous) Solution:

Set RHS = 0

$$A\frac{dh}{dt} + Kh = 0$$

assume $\underset{①}{h(t) = e^{st}} \Rightarrow \underset{②}{\frac{dh}{dt} = se^{st}}$

subbing ① & ② back into differential equation w/ RHS = 0,

$$A(se^{st}) + Ke^{st} = 0$$

$$\cancel{e^{st}}(As + K) = 0$$

$$As + K = 0 \longrightarrow \therefore s = -\frac{K}{A}$$

now, sub solution for s back into ① (the assumed solution for h(t) along with an unknown coefficient C_1

$$\Rightarrow h_{trans}(t) = C_1 e^{(-K/A)t} = C_1 e^{-(K/A)t}$$

(2) Steady-State (particular) Solution:

Suppose $Q_{in} = B$ ... constant

$$\Rightarrow A\frac{dh_{ss}}{dt} + Kh_{ss} = B$$

assume $h_{ss}(t) = E$ ... undetermined coefficient

$$\Rightarrow \frac{dh_{ss}}{dt} = 0$$

subbing into D.E.

$$\Rightarrow A(0) + KE = B$$

$$\Rightarrow E = B/K = h_{ss}(t)$$

(3) <u>Total Solution</u>: $h(t) = h_{trans}(t) + h_{ss}(t)$

$$h(t) = C_1 e^{-(K/A)t} + B/K$$

(4) <u>Initial Condition</u>: suppose $h(0) = 0$ (no water in bucket)

$$\Rightarrow h(0) = C_1 e^0 + B/K = 0 \qquad \therefore C_1 = -B/K$$

<u>Total Solution</u>: $\boxed{h(t) = -\frac{B}{K} e^{-(K/A)t} + B/K}$

<u>NOTE</u>: as $t \to \infty$, $h(t) \longrightarrow B/K$

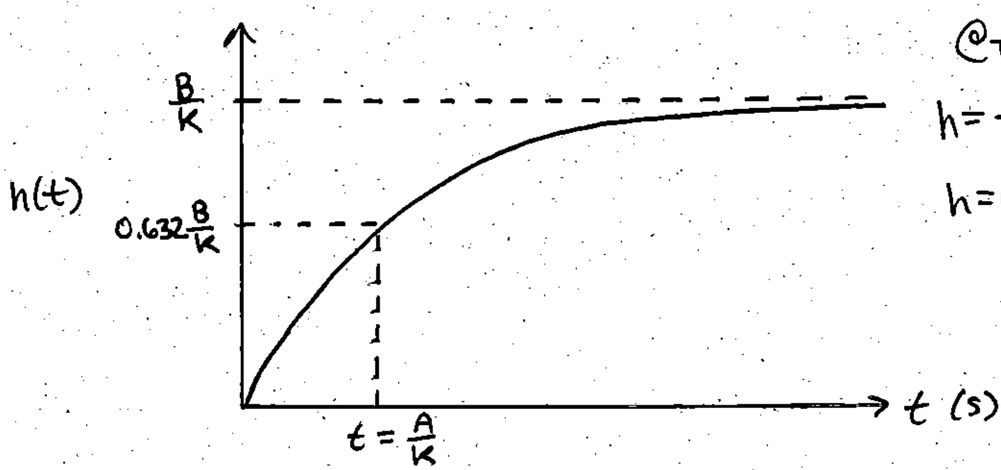
(steady-state solution)

At steady-state, h is <u>constant</u>.

Physically, the bucket continues to fill until the pressure is great enough that $Q_{out} = Q_{in} \Rightarrow \frac{dV}{dt} = 0$!

That value depends only on $B/K \equiv Q_{in}/K$

K... function of the fluid properties and outlet geometry

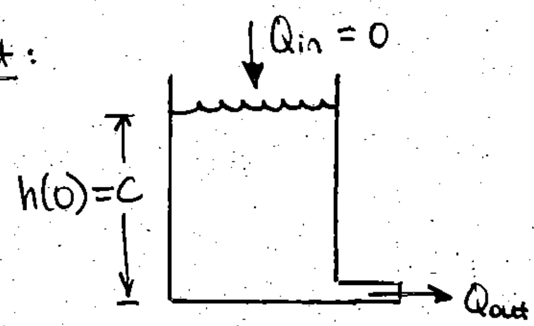


@ $t = A/K$

$h = -\frac{B}{K}\left(\frac{1}{e}\right) + \frac{B}{K}$

$h = 0.632 \frac{B}{K}$

LAB Assignment:



Difference is $Q_{in} = 0$, and $h(0) = C$.

Governing equation, $A\frac{dh}{dt} + Kh(t) = 0$

Since the right hand side is zero, $h_{ss}(t) = 0$

$\Rightarrow h(t) = h_{trans}(t) + 0 = h_{trans}(t)$

(Total solution is ONLY Transient solution!)

$\Rightarrow h(t) = C_1 e^{-(K/A)t}$ (transient sol'n same as before)

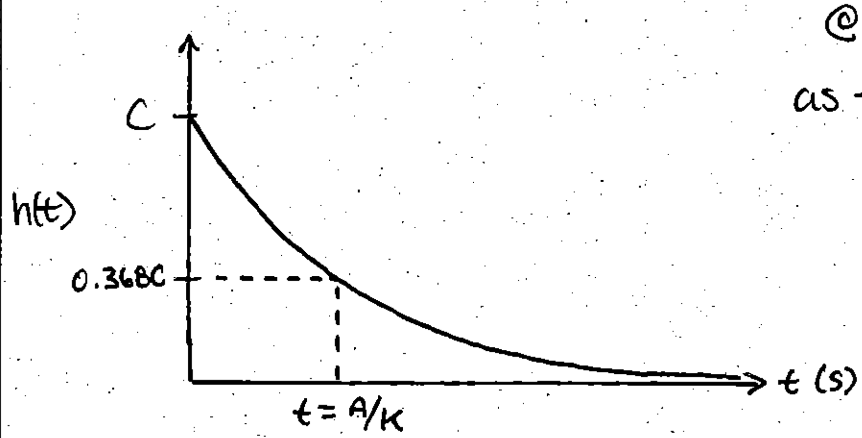
Initial Condition: $h(0) = C$

$h(0) = C_1 e^0 = C \quad \therefore C_1 = C$

$\Rightarrow h(t) = Ce^{-K/A t}$

@ $t = 0$, $h(0) = C$

as $t \to \infty$, $h(t) \to 0$



at $t = A/K$, $h(A/K) = Ce^{-1} = 0.368C$