Solving Differential Equations in Engineering

Differential equations relate an output variable $y(t)$ and its derivatives to some input function $f(t)$, i.e.,

$$A_n \frac{d^n y}{dt^n} + A_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + ... + A_1 \frac{dy}{dt} + A_0 y(t) = f(t), \tag{1}$$

where the coefficients $A_n, A_{n-1}, ...A_0$ can be constants, functions of y or functions of t. The input function $f(t)$ (also called the "forcing function") represents everything on the right hand side of the differential equation. The solution to the differential equation is the output variable, $y(t)$.

For a second order system involving position $y(t)$, velocity dy/dt and acceleration d^2y/dt^2, equation (1) takes the form

$$A_2 \frac{d^2 y}{dt^2} + A_1 \frac{dy}{dt} + A_0 y(t) = f(t). \tag{2}$$

Note that engineers often use a "dot" notation when referring to derivatives with respect to time, i.e., $\dot{y} \equiv dy/dt$, $\ddot{y} = d^2y/dt^2$, etc. In this case, equation (2) can be written as

$$A_2 \ddot{y} + A_1 \dot{y} + A_0 y(t) = f(t). \tag{3}$$

In many engineering applications, the coefficients $A_n, A_{n-1}, ...A_0$ are constants (not functions of y or t). For example, in the case of a spring-mass system subjected to an applied force $f(t)$, the governing differential equation is

$$m\ddot{y} + ky = f(t), \tag{4}$$

where m is the mass and k is the spring constant.

If the coefficients $A_n, A_{n-1}, ...A_0$ are functions of y or t, exact solutions can be difficult to obtain. In many cases exact solutions do not exist, and the solution $y(t)$ must be obtained numerically (e.g., using the differential equation solvers in MATLAB). However, in the case of *constant* coefficients, the solution $y(t)$ to a differential equation of *any order* can be obtained by following the step-by-step procedure outlined below.

1. Find the Transient Solution, $y_{trans}(t)$ (also called the "Homogeneous" or "Complementary" Solution):

a) Set the forcing function $f(t)$=0. This makes the right hand side of the equation zero:

$$A_n \frac{d^n y}{dt^n} + A_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + ... + A_1 \frac{dy}{dt} + A_0 y(t) = 0$$

b) Assume a solution of the form $y(t) = e^{st}$, and substitute it into the above equation. Note that $dy/dt = se^{st}$, $d^2y/dt^2 = s^2e^{st}$, etc., so that each term will contain an e^{st}. Since the right hand side of the equation is zero, canceling the e^{st} will result in a polynomial in s,

$$A_n s^n + A_{n\text{-}1} s^{n-1} + ... + A_1 s + A_0 = 0.$$

c) Solve for the roots of the above polynomial. These are the n values of s that make the polynomial equal to zero. Call these values s_1 , s_2 , ... s_n .

d) For the case of n distinct roots, the transient solution of the differential equation has the general form

$$y_{trans}(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + ... + c_n e^{s_n t},$$

where the constants $c_1, c_2, ..., c_n$ are determined later from the initial conditions on the problem.

e) For the special case of *repeated* roots (i.e., two of the roots are the same), the solution can be made general by multiplying one of the roots by t. For example, for a second order system with $s_1 = s_2 = s$, the transient solution is

$$y_{trans}(t) = c_1 e^{st} + c_2 t e^{st}.$$

2. Find the Steady State Solution $y_{ss}(t)$ (also called the Particular Solution):

The steady-state solution can be found using the *Method of Undetermined Coefficients*:

a) Assume (guess) the form of the steady-state solution $y_{ss}(t)$. This will usually have the same general form as the forcing function and its derivatives, but will contain unknown constants (i.e., undetermined coefficients). Example guesses are shown in the table below, where K, A, B and C are constants:

If input $f(t)$ is	Assume $y_{ss}(t)$
K	A
Kt	$A\,t + B$
Kt^2	$A\,t^2 + B\,t + C$
$K \sin\omega t$ or $K \cos\omega t$	$A \sin\omega t + B \cos\omega t$

b) Substitute the assumed steady state solution $y_{ss}(t)$ and its derivatives into the *original* differential equation.

c) Solve for the unknown (undetermined) coefficients (A, B, C, etc.). This can usually be done by equating the coefficients of like terms on the right and left hand sides of the equation.

3. **Find the Total Solution, $y(t)$:**

a) The total solution is just the sum of the transient and the steady state solutions,

$$y(t) = y_{trans}(t) + y_{ss}(t).$$

b) Apply the *initial conditions* on $y(t)$ and its derivatives. A differential equation of order n must have exactly n initial conditions, which will result in an n x n system of equations for the n constants $c_1, c_2, ..., c_n$.