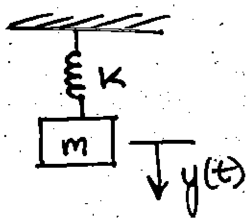
Differential Equations in Mechanical Systems

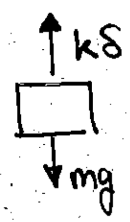
<u>Differential Equations in Mechanical Systems</u>: (ME2210, BME 3212, ME4140, ME4210)

<u>EXAMPLE</u>: Free <u>vibration</u> of a <u>Spring-Mass</u> system:



... from <u>equilibrium</u> position

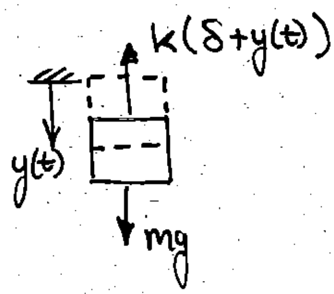
(i) Free-body Diagram @ equilibrium ($y=0$)



δ = equilibrium elongation of spring

$\underline{\Sigma F_y = 0}$: $k\delta = mg \Rightarrow \delta = \dfrac{mg}{k}$

(ii) Free-body Diagram @ $y > 0$ (mass is now <u>in</u> <u>motion</u>!)



$$\Sigma F_y = ma = m\ddot{y}$$

$$mg - k(\delta + y(t)) = m\ddot{y}$$

$$mg - k\delta - ky(t) = m\ddot{y}$$

but $\delta = mg/k$

$$\Rightarrow \cancel{mg} - \cancel{k}(\cancel{mg}/\cancel{k}) - ky(t) = m\ddot{y}$$

$$\boxed{m\ddot{y} + ky(t) = 0}$$

+ <u>Initial Conditions</u>

(second-order differential equation)

note — second derivative $\Rightarrow$ second order!

General Solution:

(1) Transient (homogeneous/complementary) Solution:

Assume $y = e^{st}$

$$\Rightarrow \dot{y} = se^{st} \;;\; \ddot{y} = s^2 e^{st}$$

plug into $m\ddot{y} + ky = 0$

$$\Rightarrow m(s^2 e^{st}) + k(e^{st}) = 0$$

$$\cancel{e^{st}}(ms^2 + k) = 0$$

$$\therefore s^2 = -\frac{k}{m} \Rightarrow s = \pm\sqrt{-\frac{k}{m}} = \pm j\sqrt{\frac{k}{m}} \quad (j = \sqrt{-1})$$

two roots: $s_1 = j\sqrt{\frac{k}{m}} \;;\; s_2 = -j\sqrt{\frac{k}{m}}$

$$\therefore y_{trans}(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

$$\therefore y_{trans}(t) = C_1 e^{j\sqrt{\frac{k}{m}}t} + C_2 e^{-j\sqrt{\frac{k}{m}}t}$$

(2) Steady-State: since RHS = 0, $y_{ss}(t) = 0$
(no forcing function)

(3) General Solution: $y(t) = y_{trans}(t) + y_{ss}(t)$

$$\Rightarrow y(t) = C_1 e^{j\sqrt{\frac{k}{m}}t} + C_2 e^{-j\sqrt{\frac{k}{m}}t}$$

Recall from complex #'s,

$$e^{j\theta} = \cos\theta + j\sin\theta \quad \text{(Euler's Formula)}$$

pronounced "Oiler's"!

we can simplify general solution above,

$$\Rightarrow y(t) = C_1 e^{j\sqrt{\frac{k}{m}}t} + C_2 e^{-j\sqrt{\frac{k}{m}}t}$$

$$\cos(\sqrt{k/m}\,t) + j\sin(\sqrt{\tfrac{k}{m}}t) \qquad \cos(-\sqrt{k/m}\,t) + j\sin(-\sqrt{\tfrac{k}{m}}t)$$

$$\begin{aligned}\Rightarrow y(t) &= C_1\left[\cos(\sqrt{\tfrac{k}{m}}t) + j\sin(\sqrt{\tfrac{k}{m}}t)\right] + C_2\left[\cos(-\sqrt{\tfrac{k}{m}}t) + j\sin(-\sqrt{\tfrac{k}{m}}t)\right] \\ &= C_1\left[\cos(\sqrt{\tfrac{k}{m}}t) + j\sin(\sqrt{\tfrac{k}{m}}t)\right] + C_2\left[\cos(\sqrt{\tfrac{k}{m}}t) - j\sin(\sqrt{\tfrac{k}{m}}t)\right] \\ &= \underbrace{(C_1 + C_2)}_{C_3}\cos(\sqrt{\tfrac{k}{m}}t) + \underbrace{(C_1 - C_2)j}_{C_4}\sin(\sqrt{\tfrac{k}{m}}t)\end{aligned}$$

NOTE: C_1 & C_2 must be complex conjugates for $y(t)$ to be real

$$\boxed{\therefore y(t) = C_3\cos(\sqrt{\tfrac{k}{m}}t) + C_4\sin(\sqrt{\tfrac{k}{m}}t)}$$

(3) Initial Conditions: eg. initial displacement & velocity

$$\Rightarrow y(0) = A, \quad \dot{y}(0) = 0$$

(require 2 initial conditions for 2^{ND} order D.E.)

$$y(0) = C_3\overbrace{\cos(0)}^{(1)} + C_4\overbrace{\sin(0)}^{0} = A \longrightarrow \therefore C_3 = A$$

$$\Rightarrow y(t) = A\cos(\sqrt{\tfrac{k}{m}}t) + C_4\sin(\sqrt{\tfrac{k}{m}}t)$$

$$\dot{y}(t) = -A\sqrt{\tfrac{k}{m}}\sin(\sqrt{\tfrac{k}{m}}t) + C_4\sqrt{\tfrac{k}{m}}\cos(\sqrt{\tfrac{k}{m}}t)$$

$$\dot{y}(0) = -A\sqrt{\tfrac{k}{m}}\overbrace{\sin(0)}^{0} + C_4\sqrt{\tfrac{k}{m}}\overbrace{\cos(0)}^{(1)} = 0 \rightarrow \therefore C_4 = 0$$

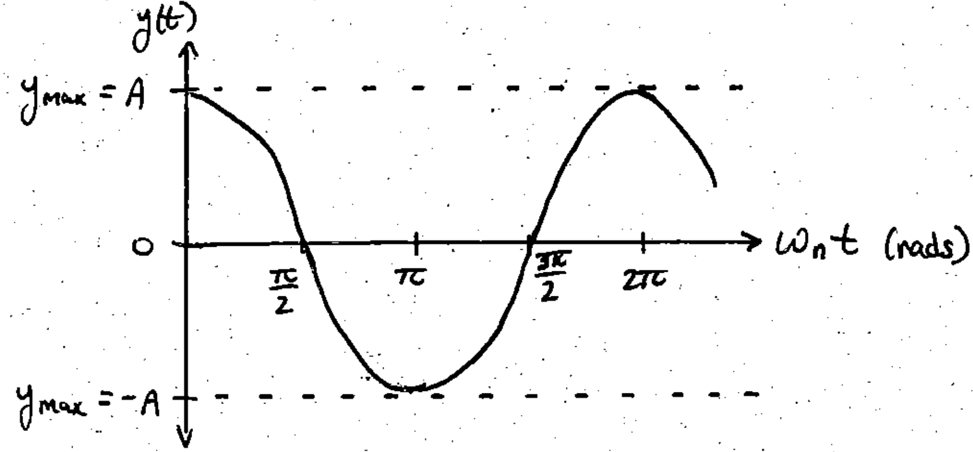
Total Solution: $y(t) = A\cos(\sqrt{\tfrac{k}{m}}t)$ OR $y(t) = A\cos(\omega_n t)$

$\omega_n = \sqrt{k/m}$ rad/s ... natural frequency of system

↳ increases with stiffness ($k\uparrow$ & $\omega_n\uparrow$)

decreases with mass ($m\uparrow$ & $\omega_n\downarrow$)

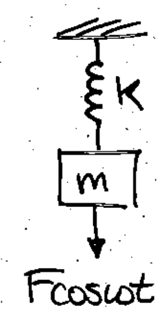
(general behavior of free vibrating system)



⇒ What if the mass is subjected to an applied force $f(t)$?
(e.g. $f(t) = F\cos\omega t \equiv$ forced vibration)

<u>Governing Equation</u>:

$$m\ddot{y} + ky(t) = F\cos\omega t$$



(1) <u>Transient Solution</u>: (unchanged!)

$$y_{trans}(t) = C_3 \cos\sqrt{\frac{k}{m}}t + C_4 \sin\sqrt{\frac{k}{m}}t$$

(2) <u>Steady-state solution</u>: $m\ddot{y} + ky = F\cos\omega t$

Recall that the steady-state solution has the same form as the forcing function!

Since the forcing function $f(t)$ is $f(t) = F\cos\omega t$ (sinusoidal)

we assume $y_{ss}(t) = A\sin\omega t + B\cos\omega t$ (also sinusoidal)

$$\Rightarrow \dot{y}_{ss}(t) = A\omega\cos\omega t - B\omega\sin\omega t$$

$$\Rightarrow \ddot{y}_{ss}(t) = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$$

Plugging back into differential equation:

$$m(-A\omega^2\sin\omega t - B\omega^2\cos\omega t) + k(A\sin\omega t + B\cos\omega t) = F\cos\omega t$$

We need to solve now for constants A & B, grouping terms,

$$(Ak - Am\omega^2)\underbrace{\sin\omega t} + (Bk - Bm\omega^2)\underbrace{\cos\omega t} = F\underbrace{\cos\omega t}$$

Comparing coefficients on $\sin\omega t$ & $\cos\omega t$ on both sides,

$\sin\omega t$: $Ak - Am\omega^2 = 0$ $\quad\therefore A = 0$

$\cos\omega t$: $Bk - Bm\omega^2 = F$

$$\therefore B = \frac{F}{k - m\omega^2}$$

$$\therefore y_{ss} = A\sin\omega t + B\cos\omega t = 0 + \left(\frac{F}{k-m\omega^2}\right)\cos\omega t$$

$$\therefore y_{ss} = \frac{F}{k-m\omega^2}\cos\omega t$$

(3) Total Solution: $y(t) = y_{trans}(t) + y_{ss}(t)$

$$y(t) = C_3\cos\left(\sqrt{\tfrac{k}{m}}t\right) + C_4\sin\left(\sqrt{\tfrac{k}{m}}t\right) + \left(\frac{F}{k-m\omega^2}\right)\cos\omega t$$

(4) Initial Conditions: $y(0) = 0$ & $\dot{y}(0) = 0$

$$y(0) = C_3\cos(0) + C_4\cancel{\sin(0)}^{0} + \left(\frac{F}{k-m\omega^2}\right)\cos(0) = 0$$

$$C_3 + \left(\frac{F}{k-m\omega^2}\right) = 0 \qquad \therefore C_3 = \frac{-F}{k-m\omega^2}$$

$$\dot{y}(0) = -C_3\sqrt{\tfrac{k}{m}}\cancel{\sin(0)}^{0} + C_4\sqrt{\tfrac{k}{m}}\cos(0) + \left(\frac{F}{k-m\omega^2}\right)\omega\cancel{\sin(0)}^{0} = 0$$

$$\therefore C_4 = 0$$

$$\boxed{\therefore y(t) = \frac{F}{(k-m\omega^2)}\left(\cos\omega t - \cos\left(\sqrt{\tfrac{k}{m}}t\right)\right)}$$

What happens to $y(t)$ as $\omega \rightarrow \sqrt{\frac{k}{m}}$?

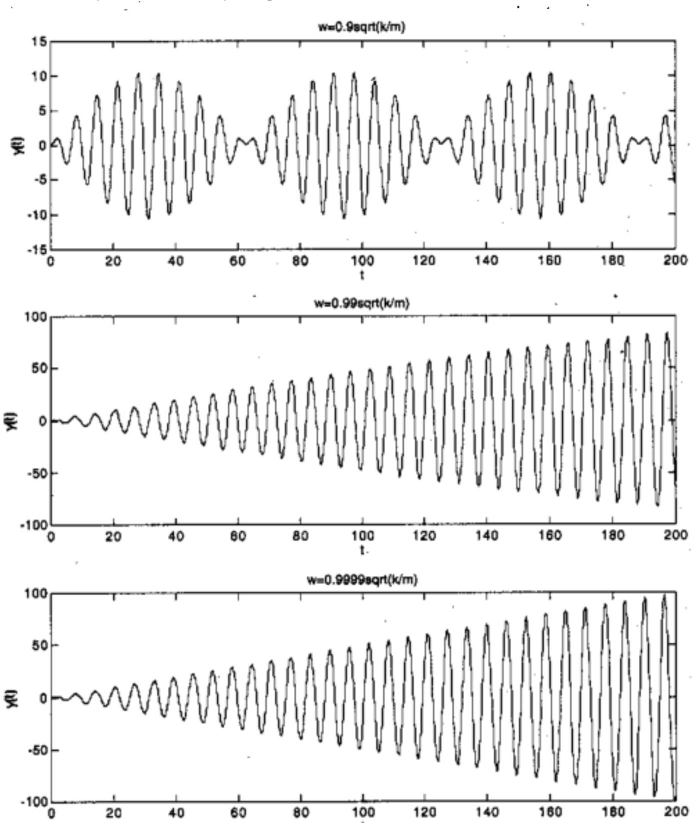
as $\omega \rightarrow \sqrt{\frac{k}{m}}$,

$$y(t) = \left(\frac{F}{0}\right)\left(\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) - \cos\left(\sqrt{\tfrac{k}{m}}\,t\right)\right) = \frac{0}{0}$$

This is an "indeterminate" form, and can be evaluated by methods of calculus not yet available to all students.

Instead, the results can be investigated by picking values of ω close to $\sqrt{k/m}$ and plotting the results.

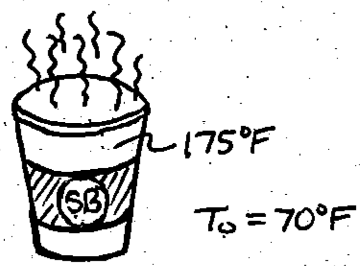
(e.g. let $k = m = F = 1$ and choose values of
$\omega = 0.9\sqrt{\frac{k}{m}}$, $0.99\sqrt{\frac{k}{m}}$, and $0.9999\sqrt{\frac{k}{m}}$)



The first plot shows the "beating" phenomenon typical of problems where forcing frequency ω is in the neighborhood of natural freq. $\omega_n = \sqrt{k/m}$.
As ω is increased to $0.99\sqrt{\frac{k}{m}}$ and $0.9999\sqrt{\frac{k}{m}}$, the last two plots show $y(t)$ increasing without bound. This is called <u>resonance</u>, and is generally undesirable!

EXAMPLE: Newton's Law of Cooling (1st ORDER D.E.)

Consider a hot coffee initially at $T(0) = 175\,°F$ sitting in a room temperature environment @ $T_\infty = 70°F$



The temperature $T(t)$ of the coffee can be approx. by Newton's Law of Cooling as:

$$\frac{dT}{dt} + kT(t) = kT_\infty$$

$T(t)$ ... temperature of coffee (°F)

T_∞ ... temp. of ambient air (°F)

k ... convective heat transfer coefficient $\left(\frac{Btu}{ft^2\,h\,°F}\right)$

Find: Temperature of coffee, $T(t)$

(1) Transient Solution: RHS = 0 , $\dot{T}(t) + kT = 0$

assume, $T_{trans}(t) = e^{st}$, $\dot{T}_{trans}(t) = se^{st}$

$$se^{st} + ke^{st} = 0$$

$$e^{st}(s+k) = 0 \qquad \therefore s = -k \qquad \boxed{\therefore T_{trans}(t) = C_1 e^{-kt}}$$

(2) Steady-State Solution: $\dot{T}(t) + kT = kT_\infty$ where $T_\infty = 70°F$

assume $T_{ss}(t) = A$ (constant) since kT_∞ = constant

$$\dot{T}_{ss}(t) = 0$$

$$0 + \cancel{k}A = \cancel{k}T_\infty \qquad \therefore A = T_\infty = 70°F$$

$$\boxed{\therefore T_{ss}(t) = 70°F}$$

(3) Total Solution: $T(t) = T_{trans}(t) + T_{ss}(t)$

$$\boxed{T(t) = C_1 e^{-kt} + 70 \ {}^\circ F}$$

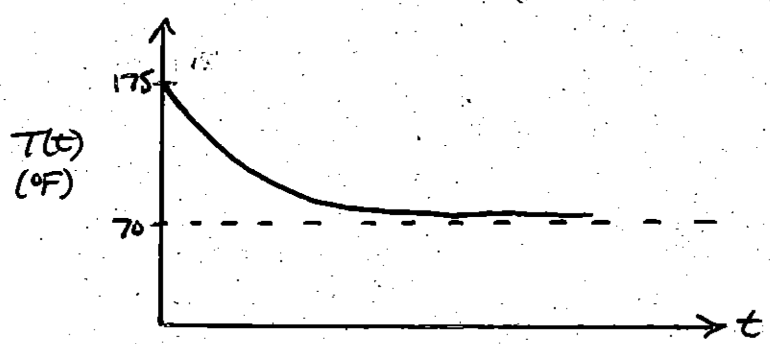
(4) Initial Conditions: $T(0) = 175^\circ F$

$T(0) = C_1 e^0 + 70 = 175^\circ \qquad \therefore C_1 = 105^\circ F$

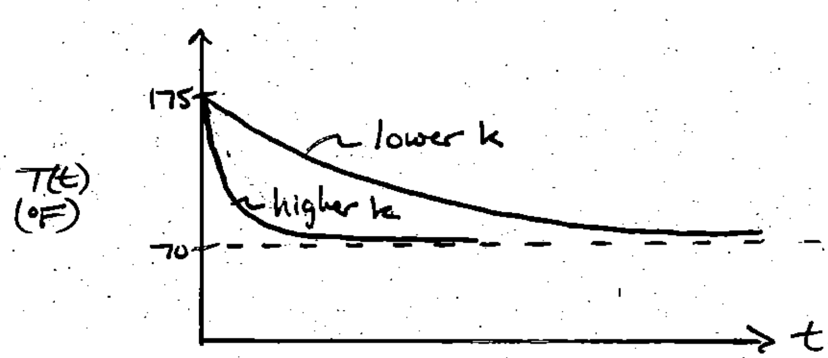
$$\boxed{\therefore T(t) = 105 e^{-kt} + 70 \ {}^\circ F}$$

$t = 0$, $T = 175^\circ F$

$t \to \infty$, $T \to 70^\circ F$ (steady-state)



What do changes in k (heat trans. coefficient) do to T(t)?



Is lower k or higher k best for a cup of coffee?