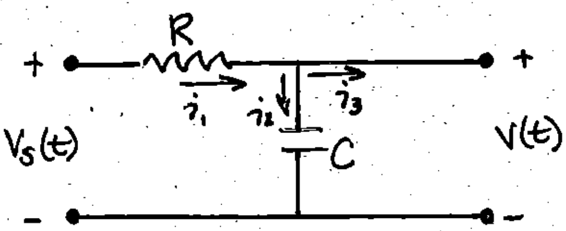
Differential Equations in Electric Systems

EXAMPLE: "Low Pass Filter"



$$i_1 = i_2 + i_3$$

$$i_1 = \frac{V_s(t) - V(t)}{R}$$

$$i_2 = C\frac{dV}{dt} = C\dot{V}(t), \quad i_3 = 0$$

Governing Equation: $\boxed{RC\dot{V} + V(t) = V_s(t)}$ ... First Order D.E.

given $V_s(t)$, we want to find the output voltage, $V(t)$

also, suppose that $V_s(t) = E\sin\omega t$, $V(0) = 0$

$$RC\dot{V} + V(t) = E\sin\omega t \quad ; \quad V(0) = 0$$

(1) Transient Solution: RHS = 0, $RC\dot{V} + V(t) = 0$

assume, $V_{trans}(t) = e^{st}$

$$\dot{V}_{trans}(t) = se^{st}$$

$$RCse^{st} + e^{st} = 0 \Rightarrow e^{st}(RCs + 1) = 0$$

$$\therefore s = -\frac{1}{RC}$$

$$\boxed{\therefore V_{trans}(t) = C_1 e^{-\frac{1}{RC}t}}$$ (dies out as $t \to \infty$)

(2) Steady-state Solution: since forcing function is sinusoid, we guess $V_{ss} = A\sin\omega t + B\cos\omega t$ (also a sinusoid)

$$\dot{V}_{ss} = A\omega\cos\omega t - B\omega\sin\omega t$$

plugging back into governing differential equation,

$$RC(A\omega\cos\omega t - B\omega\sin\omega t) + A\sin\omega t + B\cos\omega t = E\sin\omega t$$

Grouping terms,

$$(RCA\omega + B)\cos\omega t + (A - RCB\omega)\sin\omega t = E\sin\omega t$$

Comparing coefficients on $\cos\omega t$ & $\sin\omega t$ on both sides

$\cos\omega t$: $RCA\omega + B = 0 \implies B = -RC\omega A$ ... (i)

$\sin\omega t$: $A - RCB\omega = E$ ... (ii)

plugging (i) into (ii),

$$A - RC\omega(-RC\omega A) = E$$

$$A(1 + (RC\omega)^2) = E \implies A = \frac{E}{1 + (RC\omega)^2}$$

From (i), $B = -RC\omega\left(\dfrac{E}{1 + (RC\omega)^2}\right) = \dfrac{-RC\omega E}{1 + (RC\omega)^2}$

$$\boxed{\therefore V_{ss}(t) = \frac{E}{1 + (RC\omega)^2}\sin\omega t + \frac{-RC\omega E}{1 + (RC\omega)^2}\cos\omega t}$$

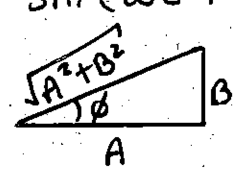
OR

$$\boxed{V_{ss}(t) = \frac{E}{1 + (RC\omega)^2}(\sin\omega t - RC\omega\cos\omega t)}$$

Recall we can further simpify addition of sinusoids if we wish as

$$A\sin\omega t + B\cos\omega t = \sqrt{A^2 + B^2}\,\sin(\omega t + \phi)$$

where, $\phi = \tan^{-1}\left(\dfrac{B}{A}\right)$



$$\Rightarrow \underbrace{\ }_{A=1}\sin\omega t \ \underbrace{-RC\omega}_{B=-RC\omega}\cos\omega t = \sqrt{1+(RC\omega)^2}\,\sin(\omega t+\phi)$$

and $\phi = \text{atan2}(-RC\omega, 1) = -\tan^{-1}\left(\frac{RC\omega}{1}\right)$

$$\Rightarrow V_{ss}(t) = \frac{E}{1+(RC\omega)^2}\left[\sqrt{1+(RC\omega)^2}\,\sin(\omega t+\phi)\right]$$

$$\boxed{\therefore V_{ss}(t) = \frac{E}{\sqrt{1+(RC\omega)^2}}\sin(\omega t+\phi), \quad \phi = -\tan^{-1}(RC\omega)}$$

(3) Total Solution: $V(t) = V_{trans}(t) + V_{ss}(t)$

$$\boxed{V(t) = C_1 e^{-\frac{1}{RC}t} + \frac{E}{\sqrt{1+(RC\omega)^2}}\sin(\omega t+\phi), \quad \phi = -\tan^{-1}(RC\omega)}$$

(4) Initial Condition: $V(0) = 0$

$$V(0) = C_1 + \frac{E}{\sqrt{1+(RC\omega)^2}}\sin\phi = 0$$

$$\therefore C_1 = -\frac{E}{\sqrt{1+(RC\omega)^2}}\sin\phi$$

But, $\sin\phi = \frac{B}{\sqrt{A^2+B^2}} = \frac{-RC\omega}{\sqrt{1+(RC\omega)^2}}$

So, $C_1 = \frac{-E}{\sqrt{1+(RC\omega)^2}}\left(\frac{-RC\omega}{\sqrt{1+(RC\omega)^2}}\right) = \frac{RC\omega E}{1+(RC\omega)^2}$

$$\boxed{\therefore V(t) = \frac{RC\omega E}{1+(RC\omega)^2}e^{-\frac{1}{RC}t} + \frac{E}{\sqrt{1+(RC\omega)^2}}\sin(\omega t+\phi) \quad (\phi = -\tan^{-1}(RC\omega))}$$

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Why is this circuit called a "low pass" filter?

Steady-State Response:

as $t \to \infty$, $V(t) \to V_{ss}(t) = \frac{E}{\sqrt{1+(RC\omega)^2}} \sin(\omega t + \phi)$

The input voltage (source) is $V_s(t) = E\sin(\omega t)$

Let us look at the ratio of the output voltage to the input voltage (only their amplitudes):

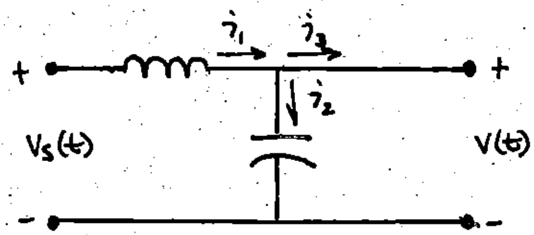
$$\frac{|V(t)|}{|V_s(t)|} = \frac{\frac{E}{\sqrt{1+(RC\omega)^2}}}{E} = \frac{1}{\sqrt{1+(RC\omega)^2}}$$

as $\omega \to \infty$, $\frac{|V(t)|}{|V_s(t)|} \to 0$

→ this circuit filters out high frequency input!

→ since only low frequencies pass through we call it a "low pass" filter!

EXAMPLE: 2^{ND} order LC circuit



$$i_1 = i_2 + i_3$$

Here, $i_2 = C\frac{dV}{dt}$, $i_3 = 0$

$$\Rightarrow i_1 = C\frac{dV}{dt} + 0 = C\frac{dV}{dt}$$

From KVL,

$$V_s(t) - V_L - V(t) = 0$$

where $V_L = L\frac{di}{dt}$

$$\Rightarrow \quad V_s(t) - L\frac{di}{dt} - V(t) = 0$$

subbing $i_1 = C\frac{dV}{dt}$,

$$V_s(t) - L\frac{d}{dt}\left(C\frac{dV}{dt}\right) - V(t) = 0$$

OR $V_s(t) - LC\frac{d^2V}{dt^2} - V(t) = 0$

Rearranging, $LC\frac{d^2V}{dt^2} + V(t) = V_s(t)$

$$\boxed{LC\ddot{V} + V(t) = V_s(t)}$$

second order DE.

Suppose $V_s(t) = E\cos\omega t$ and that there is no initial voltage or current across the capacitor

$\Rightarrow v(0) = 0, \quad i_2 = C\frac{dv}{dt}(0) = 0$

$\Rightarrow \dot{v}(0) = 0$

$$\boxed{LC\ddot{v} + v(t) = E\cos\omega t \; ; \; v(0) = 0, \; \dot{v}(0) = 0}$$

(1) Transient Solution: RHS = 0

$$LC\ddot{v} + v(t) = 0$$

Assume $V_{trans}(t) = e^{st}$

$\dot{V}_{trans}(t) = se^{st}$

$\ddot{V}_{trans}(t) = s^2e^{st}$

$LCs^2e^{st} + e^{st} = 0$

$\cancel{e^{st}}(LCs^2 + 1) = 0$

$s^2 = -\frac{1}{LC} \quad \therefore s = \pm j\sqrt{\frac{1}{LC}}$

$$\therefore V_{trans}(t) = C_1e^{j\sqrt{\frac{1}{LC}}t} + C_2e^{-j\sqrt{\frac{1}{LC}}t}$$

Via Euler's Formula,

$$\boxed{V_{trans}(t) = C_3\cos\left(\sqrt{\frac{1}{LC}}t\right) + C_4\sin\left(\sqrt{\frac{1}{LC}}t\right)}$$

(2) Steady-state Solution: $LC\ddot{v} + v(t) = E\cos\omega t$

Guess, $V_{ss}(t) = A\sin\omega t + B\cos\omega t$

$\ddot{V}_{ss}(t) = -A\omega^2\sin\omega t - B\omega^2\cos\omega t$

$LC(-A\omega^2\sin\omega t - B\omega^2\cos\omega t) + (A\sin\omega t + B\cos\omega t) = E\cos\omega t$

Grouping terms,

$(1 - LC\omega^2)A\sin\omega t + (1 - LC\omega^2)B\cos\omega t = E\cos\omega t$

$$(1-LC\omega^2)A\sin\omega t + (1-LC\omega^2)B\cos\omega t = E\cos\omega t$$

Comparing coefficients on $\sin\omega t$ and $\cos\omega t$ on both sides

$\sin\omega t$: $(1-LC\omega^2)A = 0 \quad \therefore A = 0$

$\cos\omega t$: $(1-LC\omega^2)B = E \quad \therefore B = \dfrac{E}{1-LC\omega^2}$

$$\therefore V_{ss}(t) = 0\sin\omega t + \frac{E}{1-LC\omega^2}\cos\omega t$$

$$\boxed{\therefore V_{ss}(t) = \frac{E}{1-LC\omega^2}\cos\omega t}$$

(3) Total Solution: $V(t) = V_{trans}(t) + V_{ss}(t)$

$$\boxed{\therefore V(t) = C_3\cos\left(\sqrt{\tfrac{1}{LC}}t\right) + C_4\sin\left(\sqrt{\tfrac{1}{LC}}t\right) + \frac{E}{1-LC\omega^2}\cos\omega t}$$

(4) Initial Conditions: $V(0) = 0,\ \dot{V}(0) = 0$

$$V(0) = C_3\cos(0) + \cancelto{0}{C_4\sin(0)} + \frac{E}{1-LC\omega^2}\cos(0)$$

$$C_3 + \frac{E}{1-LC\omega^2} = 0 \quad \therefore C_3 = -\frac{E}{1-LC\omega^2}$$

$$\dot{V}(t) = -\sqrt{\frac{1}{LC}}C_3\sin\left(\sqrt{\tfrac{1}{LC}}t\right) + \sqrt{\frac{1}{LC}}C_4\cos\left(\sqrt{\tfrac{1}{LC}}t\right) - \frac{E\omega}{1-LC\omega^2}\sin\omega t$$

$$\dot{V}(0) = -\sqrt{\frac{1}{LC}}C_3\cancelto{0}{\sin(0)} + \sqrt{\frac{1}{LC}}C_4\cos(0) - \frac{E\omega}{1-LC\omega^2}\cancelto{0}{\sin(0)} = 0$$

$$0 + \sqrt{\frac{1}{LC}}C_4 - 0 = 0 \quad \therefore C_4 = 0$$

$$V(t) = -\frac{E}{1-LC\omega^2}\cos\left(\sqrt{\tfrac{1}{LC}}t\right) + \frac{E}{1-LC\omega^2}\cos\omega t$$

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$$\boxed{v(t) = \frac{E}{1-LC\omega^2}\left(\cos\omega t - \cos\left(\sqrt{\tfrac{1}{LC}}\,t\right)\right)}$$

NOTE: recall spring-mass problem

$$m\ddot{y} + ky = F\cos\omega t\ , \quad y(0) = \dot{y}(0) = 0$$

$$\Rightarrow \quad y(t) = \frac{F}{k-m\omega^2}\left(\cos\omega t - \cos\left(\sqrt{\tfrac{k}{m}}\,t\right)\right)$$

Here, $LC\ddot{v} + v = E\cos\omega t$, $v(0) = \dot{v}(0) = 0$

→ Solution is identical with the following corresponding quantities

spring-mass	LC circuit
$y(t)$	$v(t)$
m	LC
k	1
F	E

We have an different physical system, but the math is exactly the same!