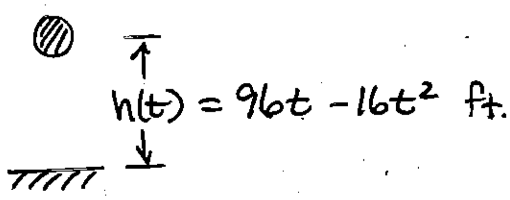
$\boxed{\text{Quadratic Equations in Engineering}}$

<u>Example</u>: (PHY 2400 - Physics, ME 2210 - Dynamics)

A projectile is fired in the vertical plane,



$h(t)$... height as a function of time (t, s)

<u>Question #1</u>: Find the time t when $h = 80$ ft.

$$\Rightarrow h(t) = 96t - 16t^2 = 80$$

or $$16t^2 - 96t + 80 = 0$$

Above is a <u>quadratic equation</u>, ie,

$$ax^2 + bx + c = 0$$

<u>Solution</u>: first divide through by 16,

$$\Rightarrow \quad t^2 - 6t + 5 = 0$$

→ Now have <u>three</u> methods to solve the quadratic equation

*<u>Method 1</u>: <u>Factoring</u>:

$$t^2 - 6t + 5 = 0$$
$$(t-1)(t-5) = 0$$
$$\boxed{\therefore t = 1 \text{ or } 5 \text{ sec}}$$

*<u>Method 2</u>: <u>Quadratic Formula</u>:

for $ax^2 + bx + c = 0$, $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here, $t^2 - 6t + 5 = 0$

ie, $a = 1$, $b = -6$, $c = 5$

$$\Rightarrow \quad t = \frac{6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)} = \frac{6 \pm \sqrt{36-20}}{2}$$

$$t = \frac{6 \pm \sqrt{16}}{2} = 3 \pm 2$$

$$\boxed{\therefore t = 1 \text{ or } 5 \text{ sec}}$$

*Method 3: Completing the Square:

$$t^2 - 6t + 5 = 0$$

$$t^2 - 6t = -5$$

$$t^2 - 6t + \left(\tfrac{-6}{2}\right)^2 = -5 + \left(\tfrac{-6}{2}\right)^2$$

$$t^2 - 6t + 9 = -5 + 9$$

$$(t-3)^2 = 4$$

$$t - 3 = \pm\sqrt{4} = \pm 2$$

$$t = 3 \pm 2 \qquad \boxed{\therefore t = 1 \text{ or } 5 \text{ sec}}$$

Plot of $h(t) = 96t - 16t^2$,

@ $t = 0$, $h(0) = 96(0) - 16(0)^2 = 0$ ft

@ $t = 1$, $h(1) = 80$ ft

@ $t = 5$, $h(5) = 80$ ft } goes up and comes back down!

What is maximum height?

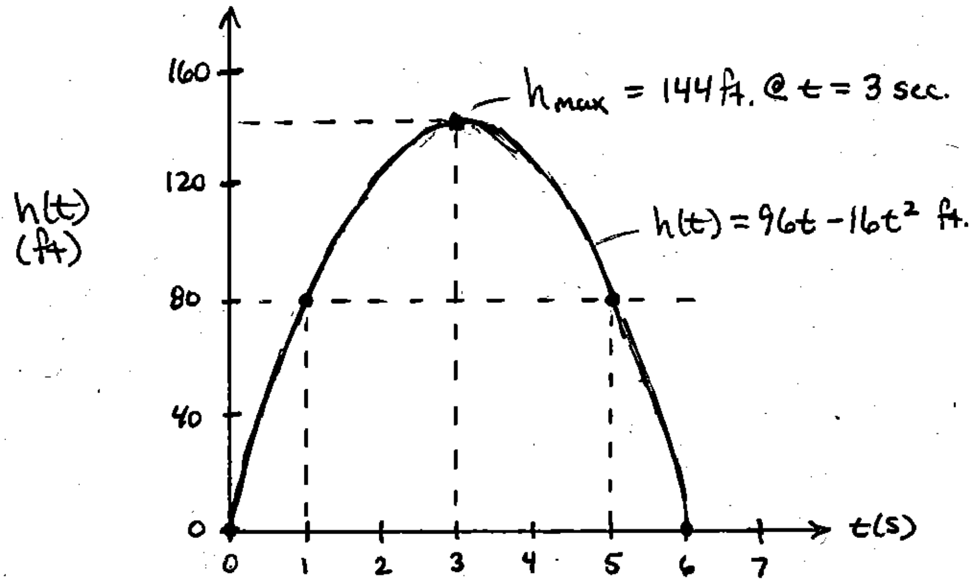
½ way between 1 & 5 seconds @ $t = 3$ s

$$h(3) = 96(3) - 16(3)^2 = 144 \text{ ft}$$

$$\boxed{h_{max} = 144 \text{ ft.}}$$

How long does it take to hit the ground? $(h=0)$

$$h(t) = 96t - 16t^2 = 0$$
$$6t - t^2 = 0$$
$$t(6-t) = 0 \qquad \boxed{\therefore t_{ground} = 6 \text{ sec}}$$



Question #2: Find time t when height $h = 144$ ft.

$$h(t) = 96t - 16t^2 = 144$$
$$16t^2 - 96t + 144 = 0$$
$$t^2 - 6t + 9 = 0$$

Factoring

$$t^2 - 6t + 9 = 0$$
$$(t-3)(t-3) = 0$$
$$t = 3,\ 3$$
$$\boxed{\therefore t = 3 \text{ sec}}$$

Quadratic Formula

$$t^2 - 6t + 9 = 0$$
$$t = \frac{6 \pm \sqrt{36-36}}{2}$$
$$t = 3 \pm 0$$
$$\boxed{\therefore t = 3 \text{ sec}}$$

Completing the Square

$$t^2 - 6t + 9 = 0$$
$$t^2 - 6t = -9$$
$$t^2 - 6t + \left(\tfrac{-6}{2}\right)^2 = -9 + \left(\tfrac{-6}{2}\right)^2$$
$$t^2 - 6t + 9 = -9 + 9$$
$$(t-3)^2 = 0$$
$$t = 3,\ 3$$
$$\boxed{\therefore t = 3 \text{ sec}}$$

Question #3: Find time t when $h = 160$ ft.

$$h(t) = 96t - 16t^2 = 160$$

$$\Rightarrow 16t^2 - 96t + 160 = 0$$

$$t^2 - 6t + 10 = 0$$

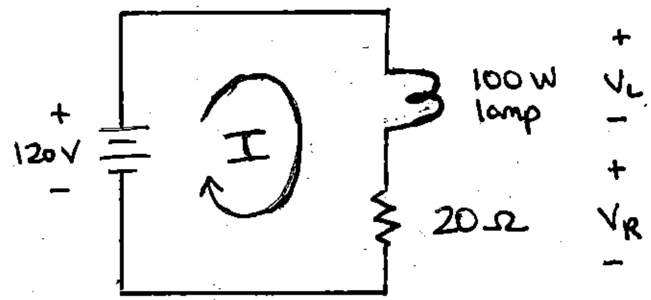
Factoring	Quadratic Formula	Completing the Square
$t^2 - 6t + 10 = 0$	$t^2 - 6t + 10 = 0$	$t^2 - 6t + 10 = 0$
CANNOT BE FACTORED!	$t = \frac{6 \pm \sqrt{36-40}}{2}$	$t^2 - 6t = -10$
	$t = \frac{6 \pm \sqrt{-4}}{2}$	$t^2 - 6t + \left(\frac{-6}{2}\right)^2 = -10 + \left(\frac{-6}{2}\right)^2$
	$t = 3 \pm \sqrt{-1}$ ↑ i or j	$t^2 - 6t + 9 = -1$
	$\boxed{t = 3 \pm j \text{ sec.}}$	$(t-3)^2 = -1$
		$(t-3) = \pm\sqrt{-1}$
		$(t-3) = \pm j$
		$\boxed{t = 3 \pm j \text{ sec}}$

→ Since the roots of the quadratic equation are complex, the projectile actually NEVER reaches 160 ft.

NOTE: we already found that $h_{max} = 144$ ft, so intuitively the math makes sense!

Example: (EE2010 - Circuits)

A 100 W (watt) lamp is connected to a 20 Ω (Ohm) resistor and a 120 V (volt) power supply



Find the current I in amperes (amps),

Solution: Using Kirchoff's Voltage Law and Ohm's Law

(KVL) $\quad 120 = V_L + V_R \quad \ldots (1)$

(Ohms) $\quad V_R = 20 I \quad \ldots (2)$

Also, for the lamp, power = voltage × current

$$\Rightarrow P_L = V_L I = 100 \text{ W}$$

$$V_L = \frac{100}{I} \quad \ldots (3)$$

plug equations (2) and (3) into (1):

$$120 = \frac{100}{I} + 20 I$$

multiply by I, $\quad 120 I = 100 + 20 I^2$

$$\Rightarrow \quad \boxed{20 I^2 - 120 I + 100 = 0}$$

We now have a quadratic equation to describe the system.

Solution: $20I^2 - 120I + 100 = 0$ (divide by 20)

$$I^2 - 6I + 5 = 0$$

Factoring	Quadratic Formula	Completing the Square
$I^2 - 6I + 5 = 0$	$I^2 - 6I + 5 = 0$	$I^2 - 6I + 5 = 0$
$(I-1)(I-5) = 0$	$I = \frac{6 \pm \sqrt{36-20}}{2}$	$I^2 - 6I = -5$
$\boxed{\therefore I = 1 \text{ or } 5 \text{ A}}$	$I = 3 \pm 2$	$I^2 - 6I + (\frac{-6}{2})^2 = -5 + (\frac{-6}{2})^2$
	$\boxed{\therefore I = 1 \text{ or } 5 \text{ A}}$	$I^2 - 6I + 9 = -5 + 9$
		$(I-3)^2 = 4$
		$I - 3 = \pm\sqrt{4}$
		$I = 3 \pm 2$
		$\boxed{\therefore I = 1 \text{ or } 5 \text{ A}}$

NOTE: 2 solutions correspond to 2 lamp choices

CASE I: $I = 1$ A

$$\Rightarrow V_L = 100/I = 100/1 = 100 \text{ volts}$$

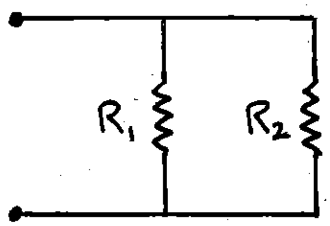
CASE II: $I = 5$ A

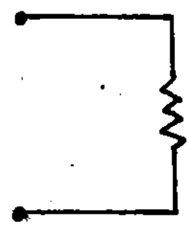
$$\Rightarrow V_L = 100/I = 100/5 = 20 \text{ volts}$$

Case I corresponds to a lamp rated at 100 volts,
Case II corresponds to a lamp rated at 20 volts

Example: (EE 2010 - circuits)

Equivalent resistance for 2 resistors in parallel,





$$R = \frac{R_1 R_2}{R_1 + R_2}$$

suppose $\quad R_1 = 4R_2 + 100\ \Omega \qquad \ldots(1)$

and $\quad R = 100\ \Omega \qquad \ldots(2)$

Find: R_1 & R_2

Solution: $\quad R = \dfrac{R_1 R_2}{R_1 + R_2}$

subbing (1) and (2) into 5

$$100 = \frac{(4R_2+100)(R_2)}{(4R_2+100) + R_2}$$

$$100 = \frac{4R_2^2 + 100R_2}{5R_2 + 100}$$

$$100(5R_2 + 100) = 4R_2^2 + 100R_2$$

$$500R_2 + 10{,}000 = 4R_2^2 + 100R_2$$

$$4R_2^2 - 400R_2 - 10{,}000 = 0$$

We now have a quadratic equation in terms of R_2,

Simplify by dividing by 4,

$$\boxed{R_2 - 100R_2 - 2500 = 0} \qquad \ldots(3)$$

Plan to solve: solve (3) for R_2 ; plug into (1) to get R_1

$$R_2^2 - 100R_2 - 2500 = 0$$

→ cannot be factored with whole numbers, lets use quadratic formula instead.

$$R_2 = \frac{100 \pm \sqrt{10{,}000 - 4(1)(-2500)}}{2}$$

$$R_2 = \frac{100 \pm \sqrt{2(10{,}000)}}{2}$$

$$R_2 = \frac{100 \pm 100\sqrt{2}}{2} = 50 \pm 50\sqrt{2}$$

Since resistance is a physical quantity that can't be negative,

$$R_2 = 50 + 50\sqrt{2} \Rightarrow R_2 = 120.7\ \Omega$$

Now from (1), $R_1 = 4R_2 + 100$

$$R_1 = 4(120.7) + 100 = 582.8\ \Omega$$

$$\boxed{\therefore R_1 = 582.8\ \Omega \text{ and } R_2 = 120.7\ \Omega}$$

→ we note from this example that not all mathematically obtained answers are physically possible (ie. negative resistance). thus we should always check our results to see if they make good physical or engineering sense!