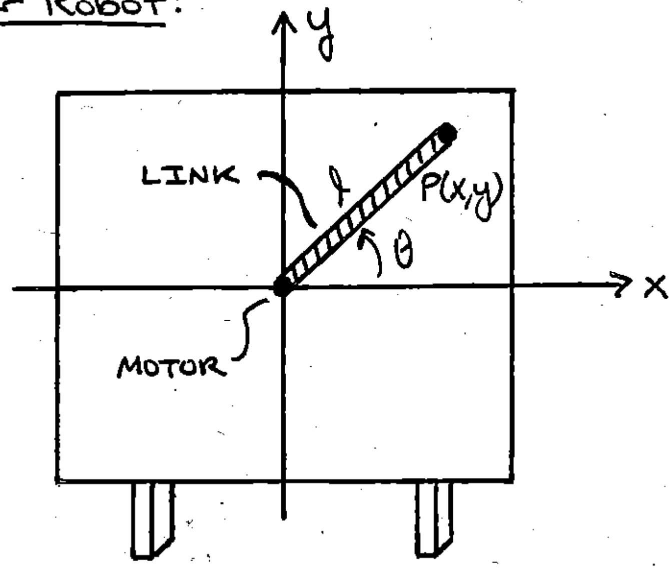
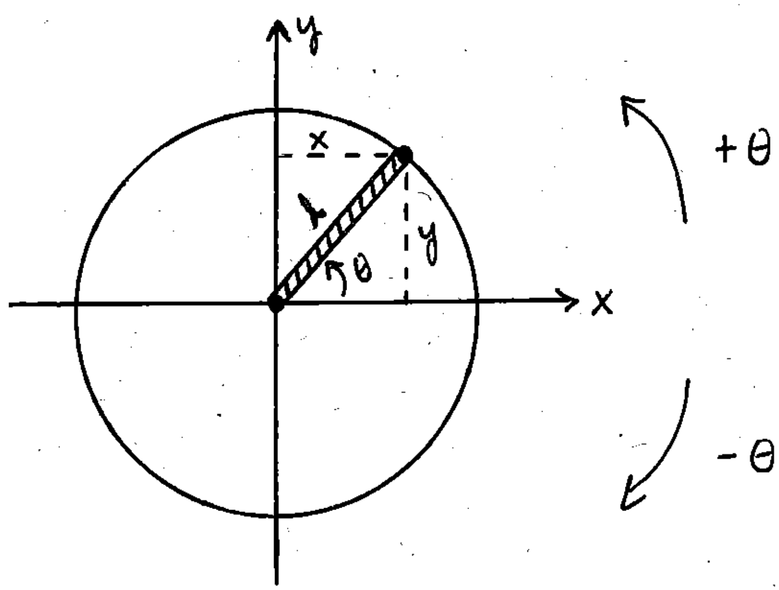
Trigonometry in Engineering

1-Link Planar Robot:



Given ℓ and θ what are the coordinates of the end point $P(x,y)$?

All points lie on a circle of radius ℓ,

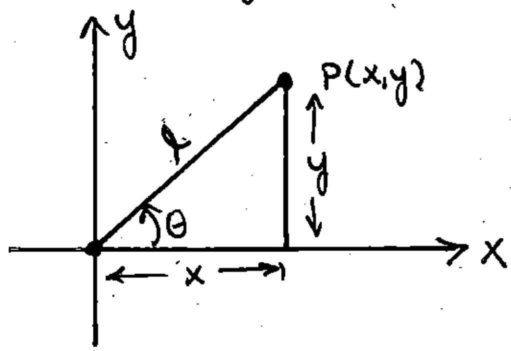


positive θ : $0 \leq \theta \leq 180°$ or $0 \leq \theta \leq \pi$ rad

negative θ : $0 \geq \theta \geq -180°$ or $0 \geq \theta \geq -\pi$ rad

NOTE: π rad $= 180°$

(x, y) related to (ℓ, θ) by reference triangle:



$\ell = \sqrt{x^2 + y^2}$ ... hypotenuse

$\cos\theta = \dfrac{x}{\ell} \Rightarrow x = \ell\cos\theta$

$\sin\theta = \dfrac{y}{\ell} \Rightarrow y = \ell\sin\theta$

NOTE: $x^2 + y^2 = (\ell\cos\theta)^2 + (\ell\sin\theta)^2$

$= \ell^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{1 \text{ (trig identity!)}}$

$\Rightarrow x^2 + y^2 = \ell^2$

or $\ell = \sqrt{x^2 + y^2}$

Other Trig Functions: $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$

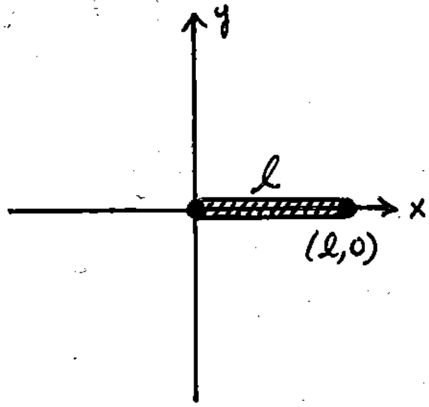
$\sec\theta = \dfrac{1}{\cos\theta}$

$\cot = \dfrac{1}{\tan\theta}$

$\csc\theta = \dfrac{1}{\sin\theta}$

<u>Example</u>: for $\theta = 0°, 90°, -90°$, and $\pm 180°$ find values of $x, y, \cos\theta$, and $\sin\theta$

$\theta = 0°$: (0 rads)



By inspection, $x = \ell$, $y = 0$

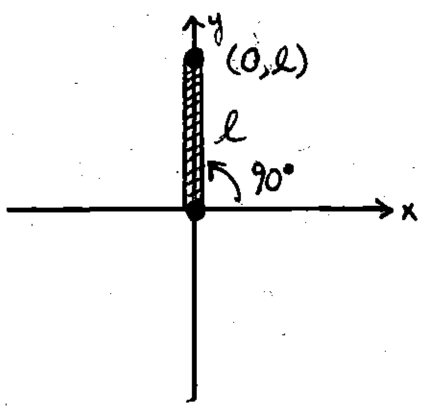
also, $x = \ell \cos 0° = \ell$

$\Rightarrow \cos 0° = 1$

$y = \ell \sin 0° = 0$

$\Rightarrow \sin 0° = 0$

$\theta = 90°$: ($\frac{\pi}{2}$ rads)



By inspection, $x = 0$, $y = \ell$

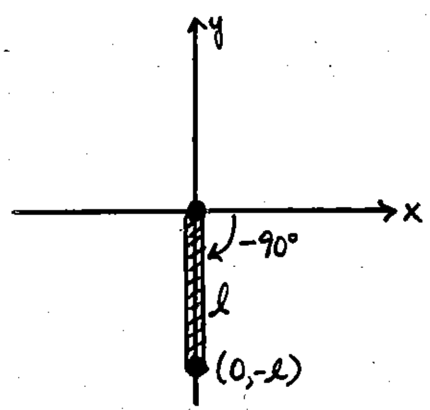
also, $x = \ell \cos 90° = 0$

$\Rightarrow \cos 90° = 0$

$y = \ell \sin 90° = \ell$

$\Rightarrow \sin 90° = 1$

$\theta = -90°$: ($-\frac{\pi}{2}$ rad)



By inspection, $x = 0$, $y = -\ell$

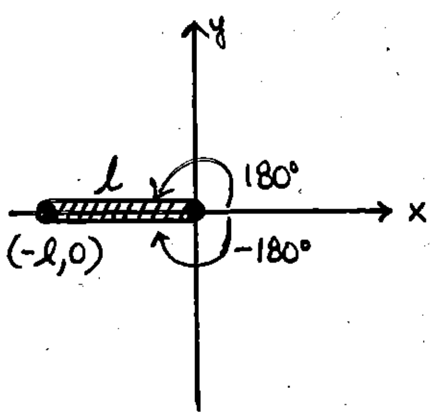
also, $x = \ell \cos(-90°) = 0$

$\Rightarrow \cos(-90°) = 0$

$y = \ell \sin(-90°) = -\ell$

$\Rightarrow \sin(-90°) = -1$

$\theta = 180°$: ($\pm\pi$ rad)



By inspection, $x = -\ell$, $y = 0$

also, $x = \ell \cos 180° = -\ell$

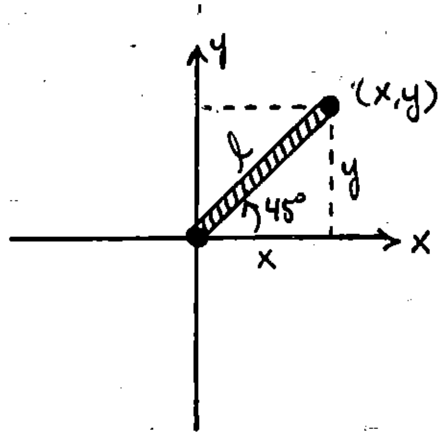
$\Rightarrow \cos 180° = -1$

$y = \ell \sin 180° = 0$

$\Rightarrow \sin 180° = 0$

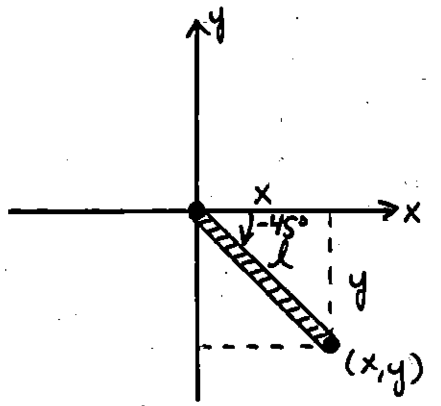
Example : find values of x & y for $\theta = 45°$, $\theta = -45°$, $\theta = 135°$, $\theta = -135°$

$\theta = 45°$:
($\frac{\pi}{4}$ rad)



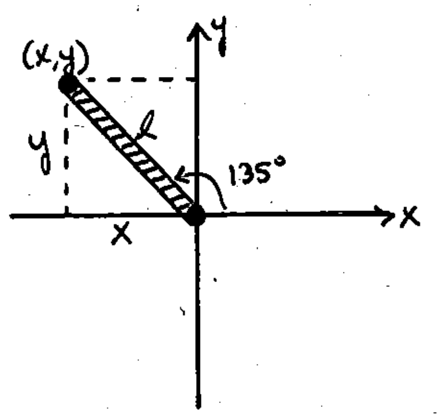
$$x = l\cos 45° = l\left(\frac{1}{\sqrt{2}}\right) = \frac{l}{\sqrt{2}}$$
$$y = l\sin 45° = l\left(\frac{1}{\sqrt{2}}\right) = \frac{l}{\sqrt{2}}$$
$$\Rightarrow (x,y) = (l/\sqrt{2},\ l/\sqrt{2})$$

$\theta = -45°$:
($-\frac{\pi}{4}$ rad)



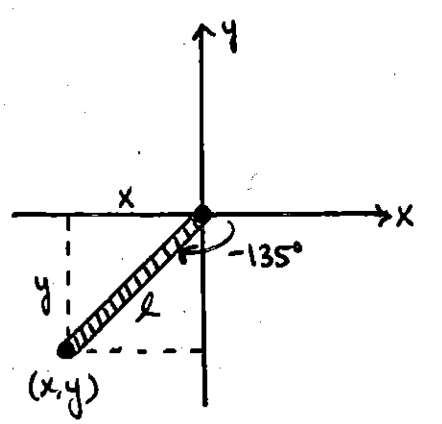
$$x = l\cos(-45°) = l\left(\frac{1}{\sqrt{2}}\right) = \frac{l}{\sqrt{2}}$$
$$y = l\sin(-45°) = l\left(-\frac{1}{\sqrt{2}}\right) = -\frac{l}{\sqrt{2}}$$
$$\Rightarrow (x,y) = (l/\sqrt{2},\ -l/\sqrt{2})$$

$\theta = 135°$:
($\frac{3\pi}{4}$ rad)



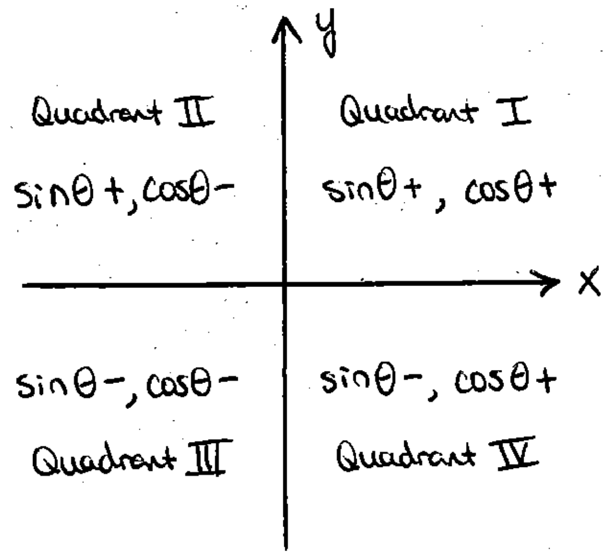
$$x = l\cos 135° = l\left(-\frac{1}{\sqrt{2}}\right) = -l/\sqrt{2}$$
$$y = l\sin 135° = l\left(\frac{1}{\sqrt{2}}\right) = -l/\sqrt{2}$$
$$\Rightarrow (x,y) = (-l/\sqrt{2},\ l/\sqrt{2})$$

$\theta = -135°$:
($-\frac{3\pi}{4}$ rad)



$$x = l\cos(-135°) = l\left(-1/\sqrt{2}\right) = -l/\sqrt{2}$$
$$y = l\sin(-135°) = l\left(-1/\sqrt{2}\right) = -l/\sqrt{2}$$
$$\Rightarrow (x,y) = (-l/\sqrt{2},\ -l/\sqrt{2})$$

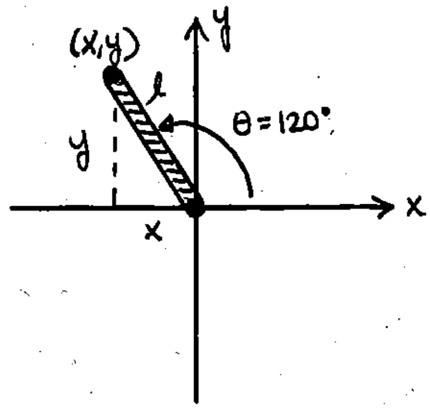
Summary: Values of $\sin\theta$, $\cos\theta$ and all other trig. functions depend on the quadrant and the reference angle:



Values of $\sin\theta$, $\cos\theta$, & $\tan\theta$ at various reference angles,

REF ANGLE (RAD)	0° (0)	30° ($\pi/6$)	45° ($\pi/4$)	60° ($\pi/3$)	90° ($\pi/2$)
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	—

Example: Find x & y for $\theta = 120°$



$$x = \ell \cos 120° = -\ell \cos 60° = -\frac{\ell}{2}$$

$$y = \ell \sin 120° = \ell \sin 60° = \frac{\sqrt{3}\,\ell}{2}$$

NOTE: we can also use trig identities, eg.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

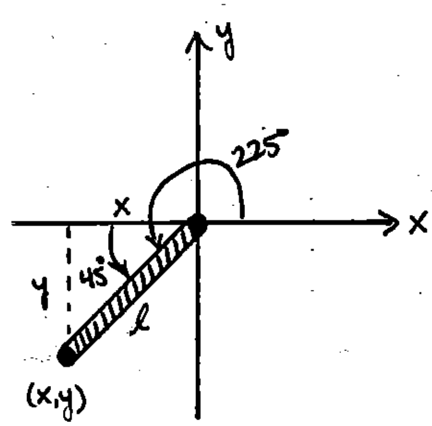
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(120°) = \sin(90°+30°) = \sin 90° \cos 30° + \cos 90° \sin 30°$$
$$= (1)(\sqrt{3}/2) + (0)(1/2) = \sqrt{3}/2$$

$$\cos(120°) = \cos(90°+30°) = \cos 90° \cos 30° - \sin 90° \sin 30°$$
$$= (0)(\sqrt{3}/2) - (1)(1/2) = -1/2$$

$$\Rightarrow x = \ell \cos 120° = \ell\left(-\tfrac{1}{2}\right) = -\ell/2$$
$$y = \ell \sin 120° = \ell(\sqrt{3}/2) = \frac{\sqrt{3}\,\ell}{2}$$

Example: Find x & y for $\theta = 225°$

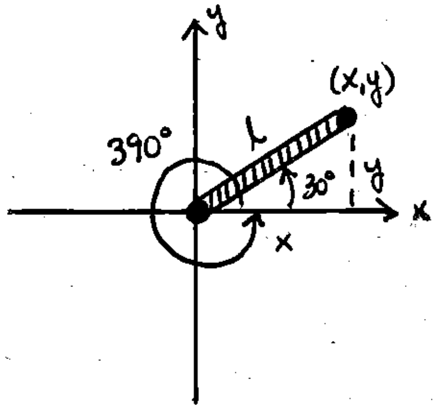


$$x = \ell \cos 225° = -\ell \cos 45° = -\frac{\ell}{\sqrt{2}}$$

$$y = \ell \sin 225° = -\ell \sin 45° = \frac{-\ell}{\sqrt{2}}$$

$$\Rightarrow (x,y) = \left(\frac{-\ell}{\sqrt{2}}, \frac{-\ell}{\sqrt{2}}\right)$$

<u>Example</u>: Find x & y for $\theta = 390°$

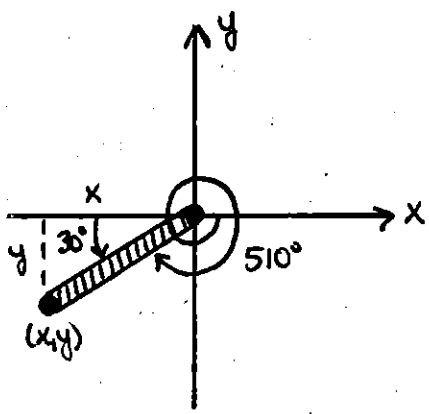


$$x = l\cos 390° = l\cos 30° = \frac{\sqrt{3}\,l}{2}$$

$$y = l\sin 390° = l\sin 30° = \frac{l}{2}$$

$$\Rightarrow (x,y) = \left(\frac{\sqrt{3}}{2}l,\ l/2\right)$$

<u>Example</u>: Find x & y for $\theta = -510°$



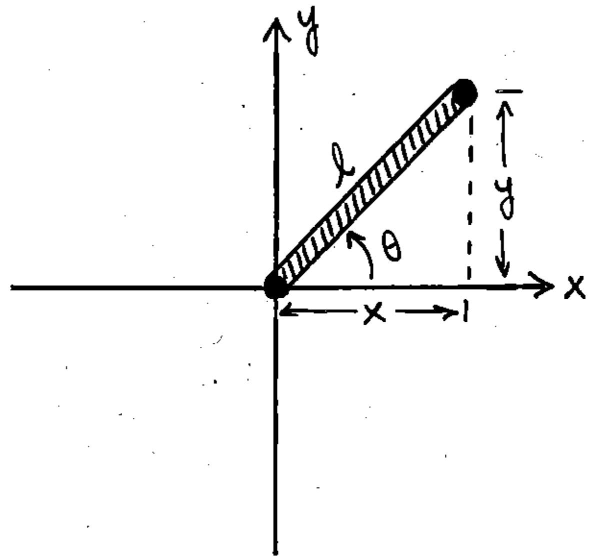
$$x = l\cos 510° = -l\cos 30° = -\frac{\sqrt{3}}{2}l$$

$$y = l\sin 510° = -l\sin 30° = -l/2$$

$$\Rightarrow (x,y) = \left(-\frac{\sqrt{3}}{2}l,\ -l/2\right)$$

The Inverse Problem:

Given x ; y, what are ℓ and θ?



We know $x = \ell\cos\theta$ and $y = \ell\sin\theta$

squaring and adding,

$$x^2 + y^2 = (\ell\cos\theta)^2 + (\ell\sin\theta)^2$$
$$= \ell^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = \ell^2$$

$$\Rightarrow \boxed{\ell = \sqrt{x^2 + y^2}}$$

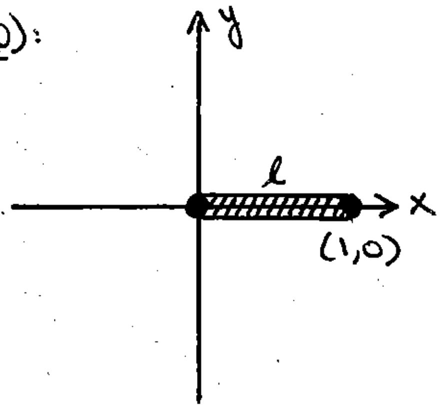
Also, $\frac{y}{x} = \frac{\ell\sin\theta}{\ell\cos\theta} = \tan\theta$

$$\Rightarrow \boxed{\theta = \tan^{-1}(y/x)}$$

NOTE: need to keep track of the signs of y and x in order to locate the quadrant

<u>Example</u>: Find ℓ and θ for the following points (x,y):

<u>$(x,y) = (1,0)$</u>:

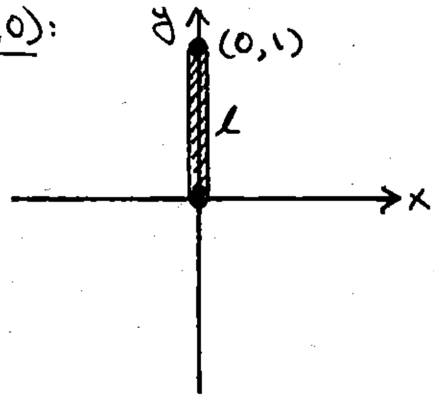


By inspection, $\ell = 1$; $\theta = 0°$

also, $\ell = \sqrt{x^2+y^2} = \sqrt{1^2+0^2} = 1$

$$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0°$$

<u>$(x,y) = (1,0)$</u>:

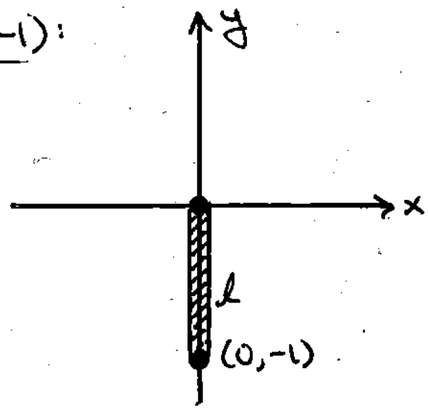


By inspection, $\ell = 1$, $\theta = 90°$

also, $\ell = \sqrt{x^2+y^2} = \sqrt{0^2+1^2} = 1$

$$\theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = 90°$$

<u>$(x,y) = (0,-1)$</u>:

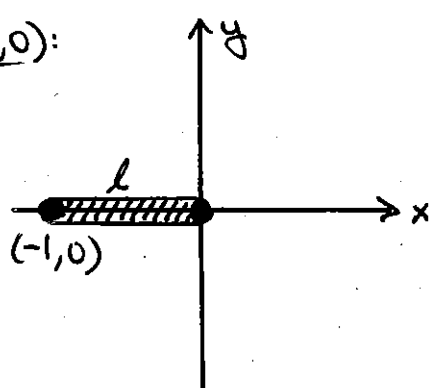


By inspection, $\ell = 1$, $\theta = -90°$

also, $\ell = \sqrt{x^2+y^2} = \sqrt{0^2+(-1)^2} = 1$

$$\theta = \tan^{-1}\left(\frac{-1}{0}\right) = \tan^{-1}(-\infty) = -90°$$

<u>$(x,y) = (-1,0)$</u>:

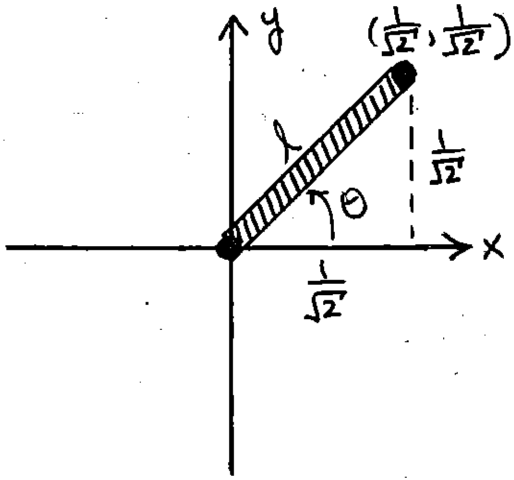


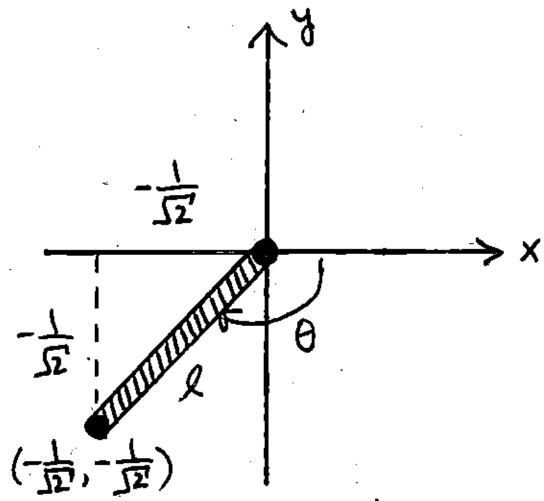
By inspection, $\ell = 1$, $\theta = 180°$

also, $\ell = \sqrt{x^2+y^2} = \sqrt{(-1)^2+0^2} = 1$

$$\theta = \tan^{-1}\left(\frac{0}{-1}\right) + 180°$$
$$= \tan^{-1}(0) + 180° = 180°$$

Consider now the following two cases,





$$(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\ell = \sqrt{x^2+y^2} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right) = 45°$$

calculator: $\theta = 45°$ ✓

$$(x,y) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$\ell = \sqrt{x^2+y^2} = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{-1/\sqrt{2}}{-1/\sqrt{2}}\right) = \tan^{-1}(1)$$

calculator: $\theta = 45°$ ... wrong!

→ calculator function $\tan^{-1}(y/x)$ returns values in the range $-90° \leq \theta \leq 90°$ only!

→ need a function that keeps track of signs of x & y (correct quadrant)

MATLAB: atan2(y,x) : four quadrant arctangent

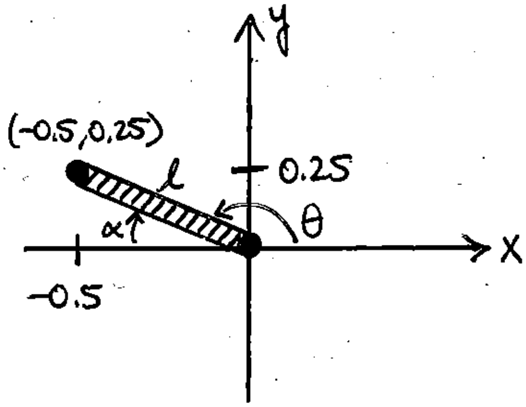
atan2(y,x) computes $\tan^{-1}(y/x)$ & keeps track of the signs of x & y

→ returns values in the range $-\pi \leq \theta \leq \pi$ rads.

case 2, $(x,y) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

MATLAB: $\text{atan2}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = (-2.3562 \text{ rad})\left(\frac{180^\circ}{\pi \text{ rad}}\right) = -135^\circ$

Example: Find ℓ and θ if $(x,y) = (-0.5, 0.25)$



$\ell = \sqrt{x^2+y^2} = \sqrt{(-0.5)^2 + (0.25)^2}$

$\boxed{\Rightarrow \ell = 0.5590}$

calculator: $\theta = \tan^{-1}\left(\frac{0.25}{-0.5}\right)$
$= \cancel{-26.57^\circ}$ → wrong!

Method 1: Ignore the signs of x ; y and use ref. angle α.

$$\theta = 180^\circ - \alpha = 180^\circ - \tan^{-1}\left(\frac{0.25}{0.5}\right)$$
$$= 180^\circ - 26.57^\circ \Rightarrow \boxed{\theta = 153.4^\circ}$$

Method 2: Use $\tan^{-1}(y/x)$ and add 180° to result,

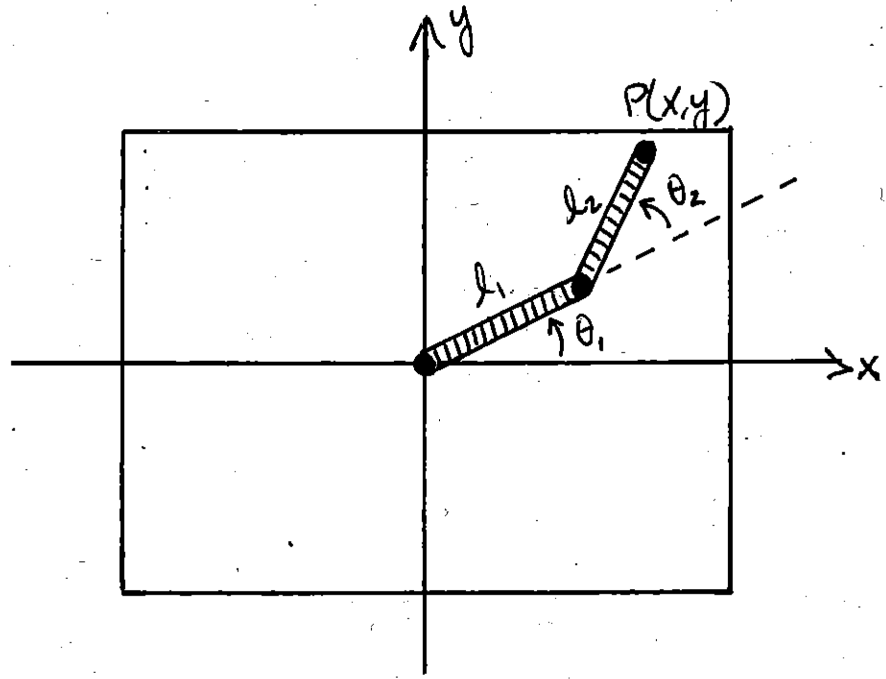
$$\theta = 180^\circ + \tan^{-1}\left(\frac{0.25}{-0.5}\right) = 180^\circ + (-26.57^\circ) = \boxed{153.4^\circ = \theta}$$

Method 3: Use atan2 (MATLAB ONLY!)

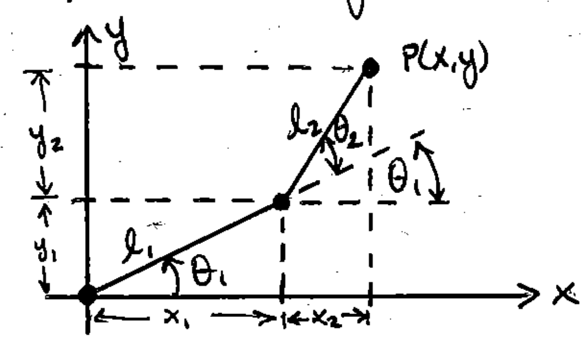
$$\theta = \text{atan2}(0.25, -0.5) = 2.6779 \text{ rads}$$

$$\theta = 2.6779 \text{ rad}\left(\frac{180^\circ}{\pi \text{ rad}}\right) \Rightarrow \boxed{\theta = 153.4^\circ}$$

Two-Link Planar Robot:



What is the position P(x,y) of the tip of the robot?



$$x = x_1 + x_2 \qquad y = y_1 + y_2$$

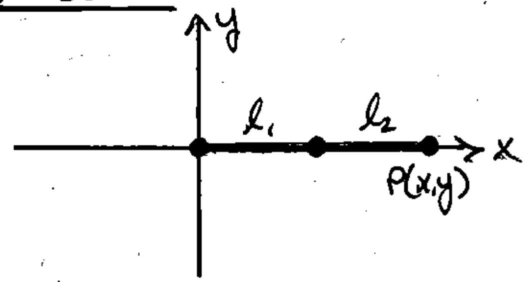
but $\quad x_1 = l_1 \cos\theta_1 \qquad x_2 = l_2 \cos(\theta_1 + \theta_2)$

$\qquad\; y_1 = l_1 \sin\theta_1 \qquad y_2 = l_2 \sin(\theta_1 + \theta_2)$

$$\boxed{\begin{aligned} x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{aligned}}$$

Example: Find the position $P(x,y)$ of the tip of the robot for the following configurations:

$\theta_1 = \theta_2 = 0°$:



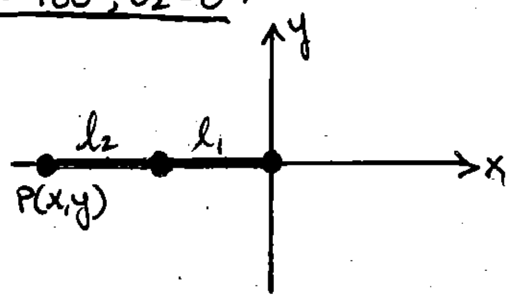
By inspection, $x = l_1 + l_2$, $y = 0$

also,

$x = l_1\cos(0) + l_2\cos(0+0) = l_1 + l_2$ ✓

$y = l_1\sin(0) + l_2\sin(0+0) = 0$ ✓

$\theta_1 = 180°, \theta_2 = 0°$:



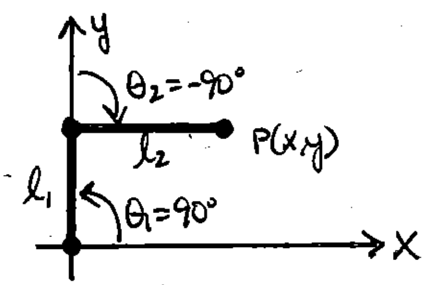
By inspection, $x = -(l_1 + l_2)$, $y = 0$

also,

$x = l_1\cos(180°) + l_2\cos(180° + 0°)$
$= l_1(-1) + l_2(-1) = -(l_1 + l_2)$ ✓

$y = l_1\sin(180°) + l_2\sin(180° + 0°)$
$= l_1(0) + l_2(0) = 0$ ✓

$\theta_1 = 90°, \theta_2 = -90°$:



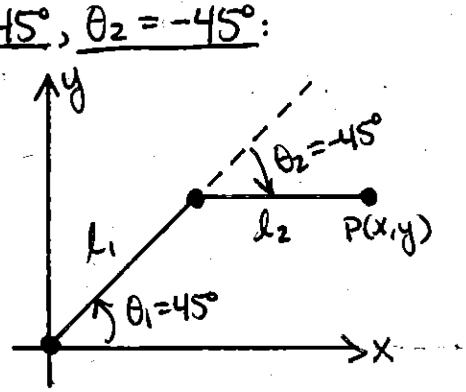
By inspection, $x = l_2$, $y = l_1$

also,

$x = l_1\cos(90°) + l_2\cos(90° + -90°)$
$= l_1(0) + l_2(1) \Rightarrow x = l_2$ ✓

$y = l_1\sin(90°) + l_2\sin(90° + -90°)$
$= l_1(1) + l_2(0) \Rightarrow y = l_1$ ✓

$\theta_1 = 45°, \theta_2 = -45°$:



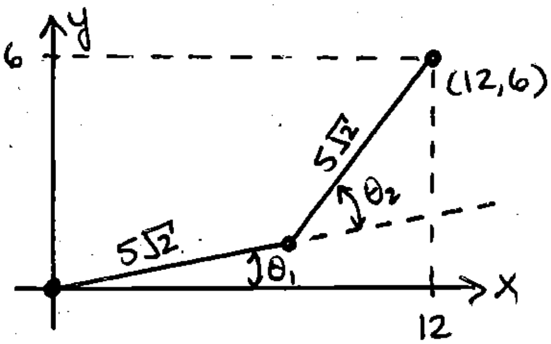
$x = l_1\cos(45°) + l_2\cos(45° + -45°)$
$= l_1(1/\sqrt{2}) + l_2(1) \Rightarrow x = l_1/\sqrt{2} + l_2$

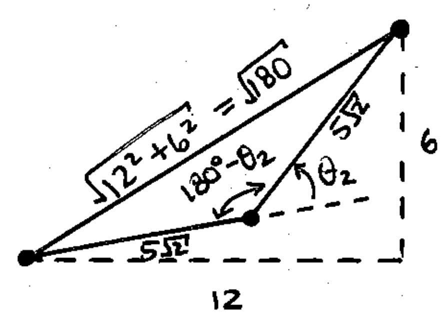
$y = l_1\sin(45°) + l_2\sin(45° + -45°)$
$= l_1\left(\frac{1}{\sqrt{2}}\right) + l_2(0) \Rightarrow y = l_1/\sqrt{2}$

Inverse Problem → Two-Link Robot:

Given: $P(x,y)$, l_1 and l_2 Find: θ_1 and θ_2

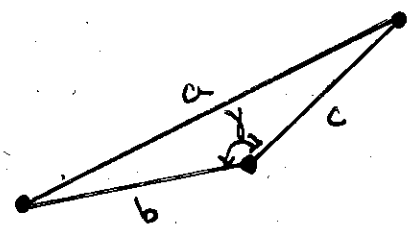
Example: $P(x,y) = (12,6)$ and $l_1 = l_2 = 5\sqrt{2}$





Recall Law of Cosines,

$$a^2 = b^2 + c^2 - 2bc\cos\gamma$$

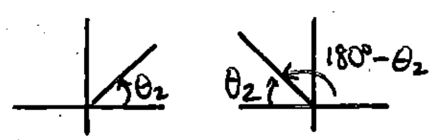


$$\Rightarrow (\sqrt{180})^2 = (5\sqrt{2})^2 + (5\sqrt{2})^2 - 2(5\sqrt{2})(5\sqrt{2})\cos(180° - \theta_2)$$

$$180 = 50 + 50 - 100\cos(180° - \theta_2)$$

$$80 = -100\cos(180° - \theta_2) \Rightarrow \cos(180° - \theta_2) = -0.8$$

NOTE: $\cos(180° - \theta_2) = -\cos(\theta_2)$

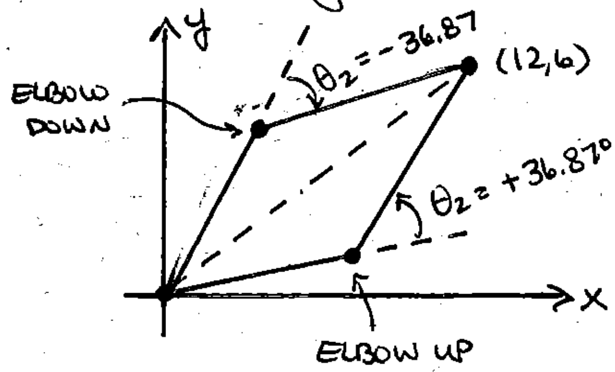


$$\Rightarrow -\cos(\theta_2) = -0.8$$

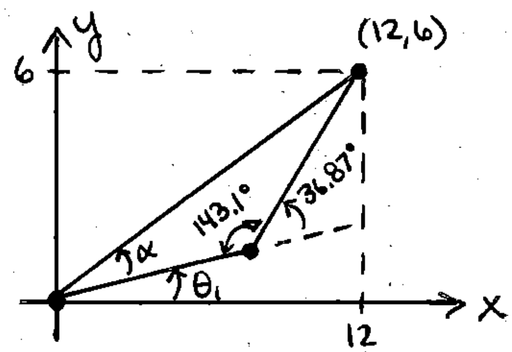
or $\cos\theta_2 = 0.8 \Rightarrow \theta_2 = \cos^{-1}(0.8)$

Two possible solutions: $\theta_2 = +36.87°$, $\theta_2 = -36.87°$

What is the meaning of two solutions for θ_2?



Elbow-Up Solution:

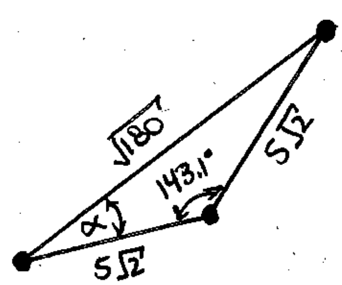


$$\tan(\alpha+\theta_1) = \frac{6}{12}$$

$$\Rightarrow \alpha + \theta_1 = \tan^{-1}\left(\frac{6}{12}\right)$$

$$\Rightarrow \alpha + \theta_1 = 26.57 \quad \ldots(1)$$

Need α: can use Law of Sines or Cosines:



Law of Sines:

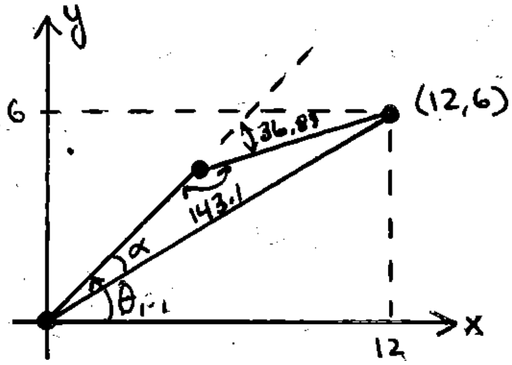
$$\frac{\sin\alpha}{5\sqrt{2}} = \frac{\sin 143.1°}{\sqrt{180}} \quad \Rightarrow \sin\alpha = \frac{5\sqrt{2}}{\sqrt{180}}\sin(143.1°) = 0.3164$$

$$\Rightarrow \alpha = \sin^{-1}(0.3164) = 18.45°$$

From (1), $\theta_1 = 26.57 - \alpha = 26.57 - 18.45 = 8.12°$

$$\boxed{\therefore \theta_1 = 8.12° \text{ and } \theta_2 = 36.9°}$$

Elbow - Down Solution:



$$\tan(\theta_1 - \alpha) = \frac{6}{12}$$

$$\Rightarrow \theta_1 - \alpha = 26.57^\circ$$

Recall from before, $\alpha = 18.45^\circ$, $\theta_1 = 26.57^\circ + 18.45^\circ = 45^\circ$

$$\boxed{\theta_1 = 45^\circ \text{ and } \theta_2 = -36.9}$$