Complex Numbers in Engineering

A complex number z has <u>real</u> and <u>imaginary</u> parts,

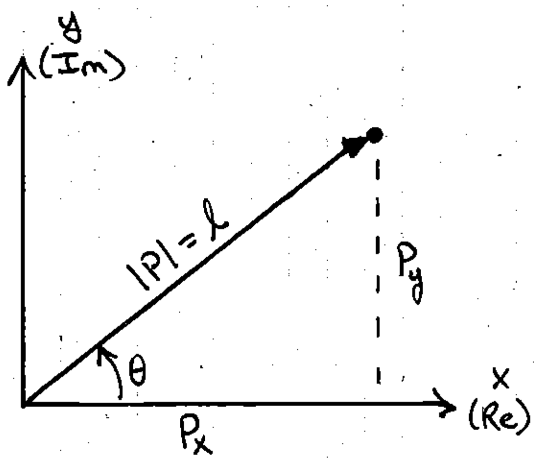
EX. $z = a + bi$
OR $z = a + bj$
OR $z = a + jb$
} all are <u>identical</u>

where $i = j = \sqrt{-1}$ ... imaginary number

also, $Re(z) = a$ ... real part
$Im(z) = b$ ... imaginary part

$\Rightarrow$ a & b are like <u>components</u> of z
(analogous to a 2-D vector)

<u>EXAMPLE</u>: Represent the position of the tip of the one-link robot as a complex number



$P = P_x + P_y j$ ... rectangular form

$Re(P) = P_x = |P|\cos\theta$ ... real part

$Im(P) = P_y = |P|\sin\theta$ ... imaginary part

Here, $|P| = \ell = \sqrt{(P_x)^2 + (P_y)^2}$ ... magnitude of P equivalently,

$$P = |P| \measuredangle \theta = \ell \measuredangle \theta \quad \text{... polar form}$$

where $\theta = \text{atan2}(P_y/P_x)$

Finally, complex numbers can be written in exponential form,

$$P = P_x + P_y j$$

$$\Rightarrow \quad P = |P|\cos\theta + j(|P|\sin\theta)$$

$$\text{OR} \quad P = |P|(\cos\theta + j\sin\theta)$$

but, $\cos\theta + j\sin\theta = e^{j\theta}$ ... Euler's Formula

NOTE: pronounced "Oiler"

$$\Rightarrow \quad P = |P|e^{j\theta} \quad \text{... exponential form}$$

Summary: The position of the tip of the 1-link robot can be described as a vector or a complex number:

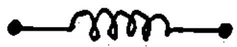
$\vec{P} = P_x\hat{i} + P_y\hat{j}$ ... vector form

$P = P_x + jP_y$ ... complex number form

$P = |P| \measuredangle \theta$ ... complex number (polar form)

$P = |P|e^{j\theta}$ ... complex number (exponential form)

EXAMPLE: Impedance of an Inductor (EE 2010 - Circuits)



L

The impedance of an inductor can be written as $Z = j\omega L$

Find Z if $L = 0.025$ H and $\omega = 2\pi f$ where $f = 60$ Hz

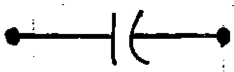
Solution:

$$Z = j\omega L = j(2\pi(60))(0.025)$$

$$Z = j\,3\pi \ \Omega$$

OR $\boxed{Z = j\,9.426\ \Omega}$

EXAMPLE: Impedance of a Capacitor (EE 2010 - Circuits)



C

The impedance of a capacitor is

$$Z = \frac{1}{j\omega C}$$

Find Z if $C = 20\mu F = 20\times10^{-6}$ F
and $f = 60$ Hz $\quad (\omega = 2\pi f = 120\pi \text{ rad/s})$

Solution:

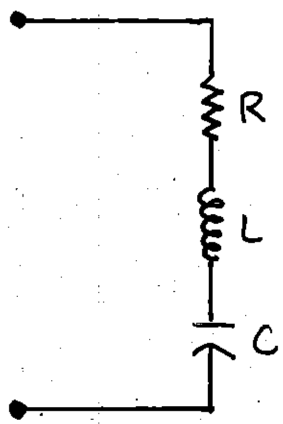
$$Z = \frac{1}{j\omega C} = \frac{1}{j(120\pi)(20\times10^{-6})} = \frac{132.6}{j}\ \Omega$$

OR $\quad Z = \frac{132.6}{j}\left(\frac{j}{j}\right) = \frac{132.6\,j}{j^2}$

BUT $\quad j^2 = \left(\sqrt{-1}\right)^2 = -1$

$$\boxed{\therefore\ Z = -132.6\,j\ \Omega}$$

EXAMPLE: Find the impedance of a resistor, inductor, and a capacitor in series:



$$Z = Z_R + Z_L + Z_C$$

where, $Z_R = R$

$Z_L = j\omega L$

$Z_C = \frac{1}{j\omega C}$

Find: Z if $R = 100\,\Omega$, $L = 0.025\,H$, $C = 20\,\mu F$, and $f = 60\,Hz$ ($\omega = 120\pi \frac{rad}{s}$)

Solution: From previous two examples,

$$Z_L = j\,9.426\,\Omega \quad \text{and} \quad Z_C = -j\,132.6\,\Omega$$

$$Z = Z_R + Z_L + Z_C = 100 + j\,9.426 + -j\,132.6$$

$$Z = 100 + j\,(9.426 - 132.6)$$

$$\boxed{Z = 100 - j\,123.2\,\Omega}$$

→ polar form: $Z = |Z| \angle \theta$

$$|Z| = \sqrt{(100)^2 + (-123.2)^2} = 158.7\,\Omega$$

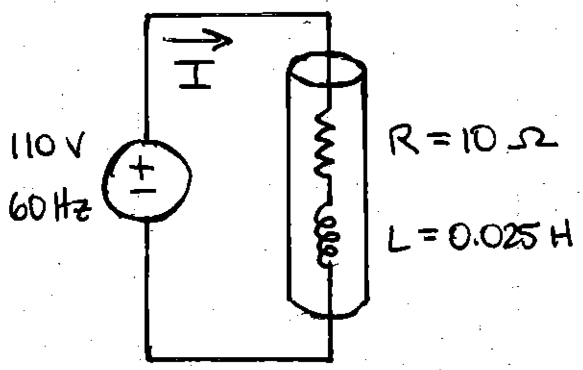
$$\theta = \text{atan2}\left(\frac{-123.2}{100}\right) = -50.93°$$

$$\boxed{Z = 158.7 \angle -50.93°\,\Omega}$$

→ exponential form: $Z = |Z| e^{j\theta} = 158.7 e^{-50.93 j}\,\Omega$

EXAMPLE:

An electric motor has a resistance $R = 10\,\Omega$ and an inductance of $L = 0.025\,H$. If the motor is connected to a 110 V, 60 Hz source, Find the current $I = V/Z$



$$Z = Z_R + Z_L$$

$$V = 110 \text{ volts}$$

Solution: $Z_R = 10\,\Omega$, $Z_L = j\omega L = j\,9.426\,\Omega$

$$\Rightarrow Z = 10 + 9.426j\ \Omega$$

$$I = \frac{V}{Z} = \frac{110}{10 + 9.426j} = \frac{110 + j0}{10 + j\,9.426}$$

* Dividing complex numbers is easier in exponential or polar form,

$$I = \frac{110 + j0}{10 + j9.426} = \frac{110 \angle 0^\circ}{\sqrt{10^2 + 9.426^2} \angle \operatorname{atan2}(9.426/10)}$$

$$I = \frac{110 \angle 0^\circ}{13.74 \angle 43.3^\circ} = \frac{110 e^{j0^\circ}}{13.74 e^{j43.3^\circ}}$$

$$I = \frac{110}{13.74} e^{j(0 - 43.3^\circ)} = 8.01 e^{j(-43.3^\circ)} = \boxed{8.01 \angle -43.3^\circ\ A}$$

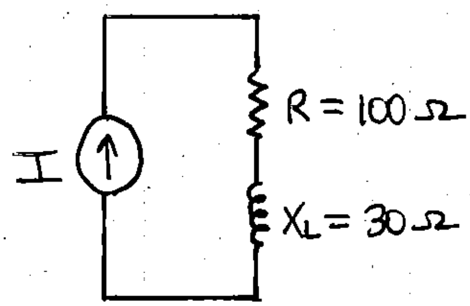
Rectangular Form: $I = 8.01 e^{-j43.3^\circ} = 8.01(\cos(-43.3) + j\sin(-43.3))$

$$\boxed{I = 5.82 - j\,5.49\ A}$$

<u>NOTE</u>: in general, $e^{j\theta} = \cos\theta + j\sin\theta$ requires that θ be in radians. However converting to radians is unnecessary for the purpose of multiplying and dividing complex numbers.

<u>EXAMPLE</u>: Find $V = IZ$ for the following circuit if

$$I = 0.1 \angle 30^\circ \text{ A}$$



$$Z = R + jX_L$$

<u>Solution</u>: $Z = R + jX_L = 100 + j30\ \Omega$

$$I = 0.1 \angle 30^\circ \text{ A}$$

want $V = IZ$

Rectangular Form

$I = 0.1 \angle 30^\circ = 0.1(\cos 30 + j\sin 30)$

$I = 0.0866 + j0.05$ A

$Z = 100 + j30\ \Omega$

$V = IZ$

$= (0.0866 + j0.05)(100 + j30)$

$= 8.66 + 2.598j + 5j + 1.5\cancel{j^2}^{(-1)}$

$\boxed{V = 7.16 + 7.598j \text{ volts}}$

OR $V = \sqrt{7.16^2 + 7.598^2} \angle \operatorname{atan2}\left(\frac{7.598}{7.16}\right)$

$\boxed{V = 10.44 \angle 46.7^\circ \text{ volts}}$

Polar Form

$I = 0.1 \angle 30^\circ = 0.1e^{j30^\circ}$

$Z = 100 + j30\ \Omega$

$= \sqrt{100^2 + 30^2} \angle \operatorname{atan2}\left(\frac{30}{100}\right)$

$Z = 104.4 \angle 16.7^\circ = 104.4e^{j16.7^\circ}$

$V = IZ$

$= (0.1 \angle 30^\circ)(104.4 \angle 16.7)$

$= (0.1e^{j30^\circ})(104.4e^{j16.7^\circ})$

$= (0.1)(104.4) \angle (30 + 16.7)$

$= (0.1)(104.4)e^{j(30+16.7)^\circ}$

$\boxed{V = 10.44 \angle 46.7^\circ \text{ volts}}$

NOTES:

- Adding complex numbers is easist in rectangular form:
  (just add real and imaginary parts)
  $$Z_1 = Re_1 + j\,Im_1 \;;\; Z_2 = Re_2 + j\,Im_2$$
  $$Z_1 \pm Z_2 = (Re_1 \pm Re_2) + j\,(Im_1 \pm Im_2)$$

- Multiplying / Dividing complex numbers is easist in polar form:
  (just multiply / divide magnitudes and add / subtract angles)
  $$Z_1 = M_1 \angle \theta_1 \;;\; Z_2 = M_2 \angle \theta_2$$
  $$Z_1 \cdot Z_2 = (M_1 M_2) \angle (\theta_1 + \theta_2)$$
  $$\frac{Z_1}{Z_2} = \frac{M_1}{M_2} \angle (\theta_1 - \theta_2)$$

Complex Conjugates:

The complex conjugate of $Z = a + jb$ is

$$Z^* = a - jb$$

NOTE: $Z \cdot Z^* = |Z|^2 = a^2 + b^2$

ie, $Z \cdot Z^* = (a + jb)(a - jb) = a^2 - (jb)^2$

$$= a^2 - (-1)b^2 = a^2 + b^2$$

$$|Z|^2 = \left(\sqrt{a^2 + b^2}\right)^2 = a^2 + b^2$$

$$\Rightarrow \boxed{Z \cdot Z^* = |Z|^2}$$

EX: If $C = 3 + j4$, find $C \cdot C^*$ using rectangular and polar forms

$$C = 3 + j4 \quad \Rightarrow \quad C^* = 3 - j4$$

$$C \cdot C^* = (3+j4)(3-j4) = (3)^2 - (j4)^2$$
$$= 9 - -16 = 25$$

$$\boxed{C \cdot C^* = 25}$$

<u>NOTE</u>: $(|C|)^2 = \left(\sqrt{3^2+4^2}\right)^2 = 25$ ✓

<u>Polar Form</u>:

$$C = 3 + j4 = \sqrt{(3)^2+(4)^2} \angle \operatorname{atan2}(4/3)$$
$$C = 5 \angle 53.1°$$

$$C^* = 3 - j4 = \sqrt{(3)^2+(-4)^2} \angle \operatorname{atan2}(-4/3)$$
$$C^* = 5 \angle -53.1°$$

$$C \cdot C^* = (5\angle 53.1°)(5\angle -53.1°)$$
$$= (5)(5) \angle (53.1° - 53.1°)$$
$$= 25 \angle 0° = 25 \ ✓$$

again, $\boxed{C \cdot C^* = 25}$

<u>NOTE</u>: when we switch the sign on the <u>imaginary</u> part of a complex number in rectangular form to get the complex conjugate, this flips the complex number over the <u>real axis</u> (switches the sign on the angle in polar form!)