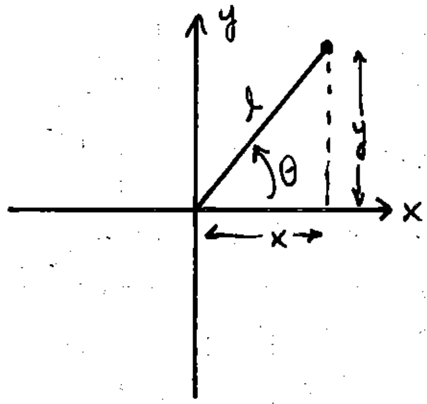
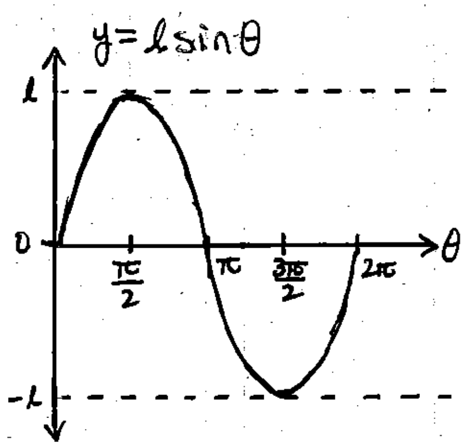
Sinusoids & Harmonics in Engineering

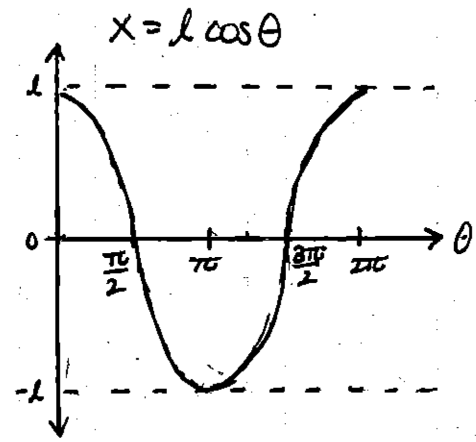
Plot the x and y components of the one-link planar robot for $0 \leq \theta \leq 2\pi$:



$x = \ell \cos\theta$

$y = \ell \sin\theta$





Both $y = \ell \sin\theta$ and $x = \ell\cos\theta$ are <u>periodic</u>:

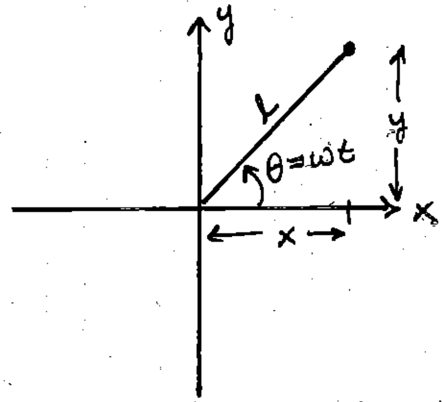
ie $\sin(\theta + 2\pi) = \sin\theta$

$\cos(\theta + 2\pi) = \cos(\theta)$

→ Repeat every 2π radians (period $T = 2\pi$)

→ amplitude, ℓ ... maximum / minimum values

Suppose now the robot rotates with an angular frequency ω:



$$y = l\sin\theta = l\sin\omega t$$

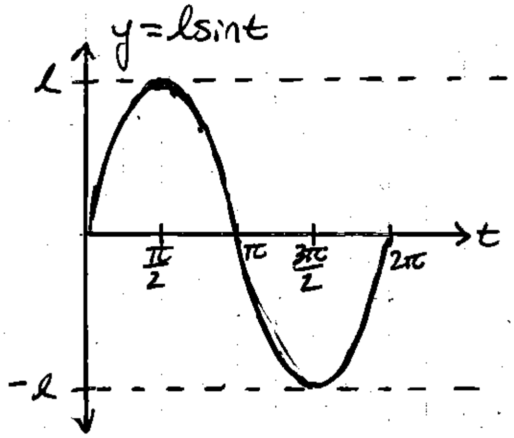
$$x = l\cos\theta = l\cos\omega t$$

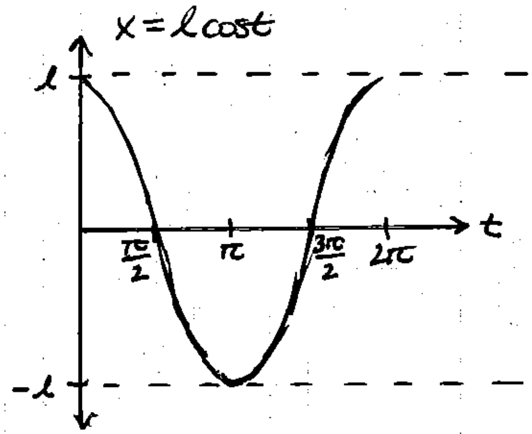
Plot the x and y components if the robot starts at $\theta = 0$ and take $t = 2\pi$ seconds to complete one revolution.

$$\theta = \omega t \quad \Rightarrow \quad \omega = \frac{\theta}{t} = \frac{2\pi \text{ rad}}{2\pi \text{ sec}} = 1 \, \frac{\text{rad}}{\text{s}}$$

$$y = l\sin\omega t = l\sin t$$

$$x = l\cos\omega t = l\cos t$$





$T = 2\pi$ seconds/cycle ... <u>period</u>

$f = \frac{1}{T} = \frac{1}{2\pi}$ cycles/sec. ... <u>frequency</u>

$\omega = 2\pi f = (2\pi)\left(\frac{1}{2\pi}\right) = 1$ rad/s ... <u>angular frequency</u>

Relations between T, f, and ω:

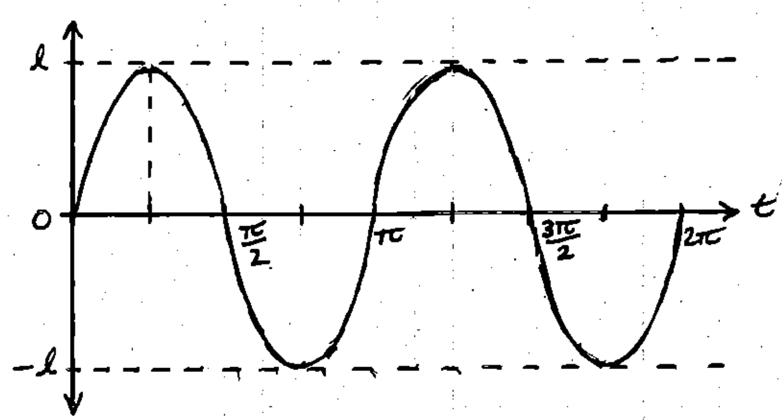
$$\boxed{\omega = 2\pi f}$$

$$f = \frac{1}{T} \Rightarrow \boxed{\omega = \frac{2\pi}{T} \quad \text{or} \quad T = \frac{2\pi}{\omega}}$$

Suppose now the frequency is doubled, ie

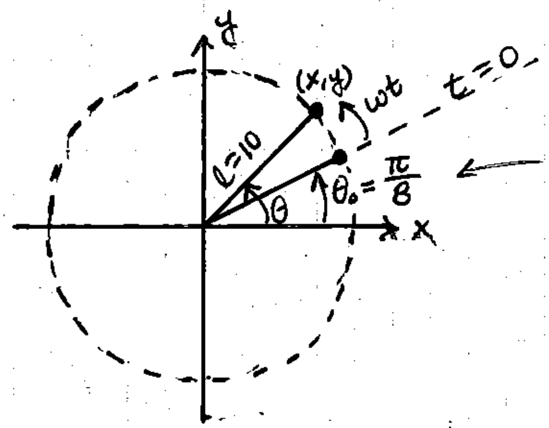
$$\omega = 2 \text{ rad/s} \quad \text{or} \quad T = \frac{2\pi}{2} = \pi \text{ sec}$$

eg. $y = \ell \sin \omega t = \ell \sin 2t$



Twice the frequency = Twice as many cycles in 2π seconds!

Suppose the robot starts from $\theta_0 = \pi/8$ rad at time $t = 0$, and takes 1 second to complete a revolution:



θ_0 = phase angle = ϕ

$x = l\cos\theta$

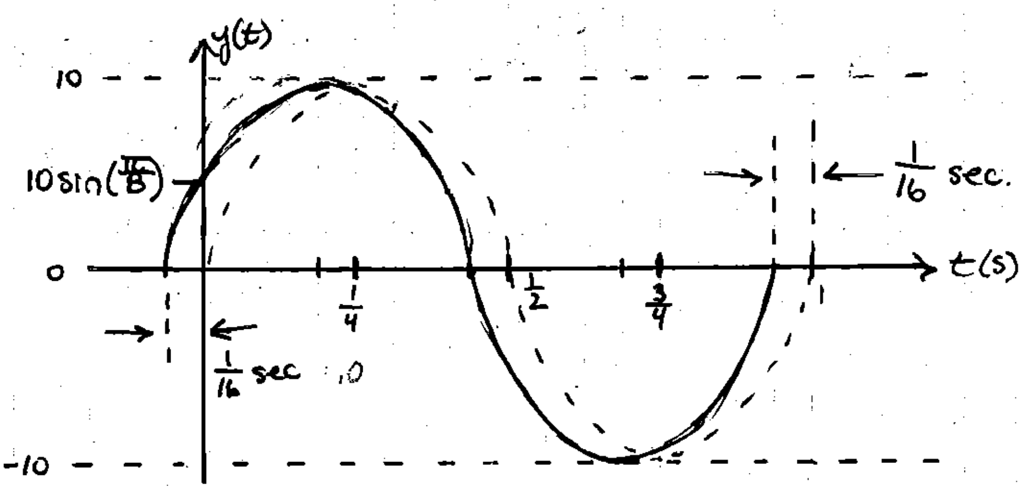
$y = l\sin\theta$

at any time t,

$$x = l\cos\theta = l\cos(\omega t + \pi/8)$$
$$y = l\sin\theta = l\sin(\omega t + \pi/8)$$

Here $l = 10$, $\omega = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$ rad/s, $\phi = \frac{\pi}{8}$ rad

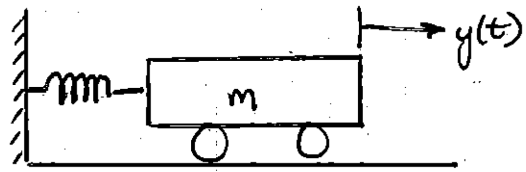
$$\Rightarrow \quad x = 10\cos(2\pi t + \pi/8)$$
$$y = 10\sin(2\pi t + \pi/8)$$



$\theta = \omega t = \phi = \frac{\pi}{8} \Rightarrow t = \pi/8 / \omega = \pi/8 / 2\pi = \frac{1}{16}$ s

$\Rightarrow$ axes shifted by $t = \frac{1}{16}$ sec. (phase shift)

<u>EX</u>: Harmonic motion of a spring-mass system (ME4210 - Dynamics)



Suppose $y(t) = 2\sin(6\pi t + \frac{\pi}{2})$ meters

(a) <u>Find</u>: amplitude, frequency, period, and phase

in general, $y(t) = A\sin(\omega t + \phi)$

A ... amplitude
ω ... angular frequency
ϕ ... phase angle

Here, $\boxed{A = 2 \text{ m}}$ $\boxed{\omega = 6\pi \text{ rad/s}}$

$\Rightarrow \omega = 2\pi f = \frac{2\pi}{T} \Rightarrow f = \frac{\omega}{2\pi} = \frac{6\pi}{2\pi} = 3$ cycles/s

$\boxed{\therefore f = 3 \text{ Hz}}$

$T = \frac{1}{f} = \frac{1}{3}$ $\boxed{\therefore T = \frac{1}{3} \text{ s}}$

phase angle, $\boxed{\phi = \frac{\pi}{2} \text{ rads.}}$

phase shift, $\phi = \frac{\pi}{2} = \omega t = 6\pi t$

$\Rightarrow t = (\frac{\pi}{2})/6\pi = \frac{1}{12}$ $\boxed{\therefore t = \frac{1}{12} \text{ sec.}}$

(b) <u>Find</u>: $y(t)$ at $t = 2.0$ seconds

$y(2) = 2\sin(6\pi(2) + \pi/2) = 2\sin(\underbrace{12\pi}_{\text{6 full revolutions}} + \frac{\pi}{2})$

$y(2) = 2\sin(\frac{\pi}{2}) \Rightarrow \boxed{y(2) = 2.0 \text{ m}}$

(c) <u>Find</u>: the time required to first reach the maximum <u>negative</u> displacement.

$$y(t) = 2\sin(6\pi t + \pi/2)$$

$$y_{max} = -2 \quad \text{at} \quad \theta = (6\pi t + \pi/2) = \frac{3\pi}{2}$$

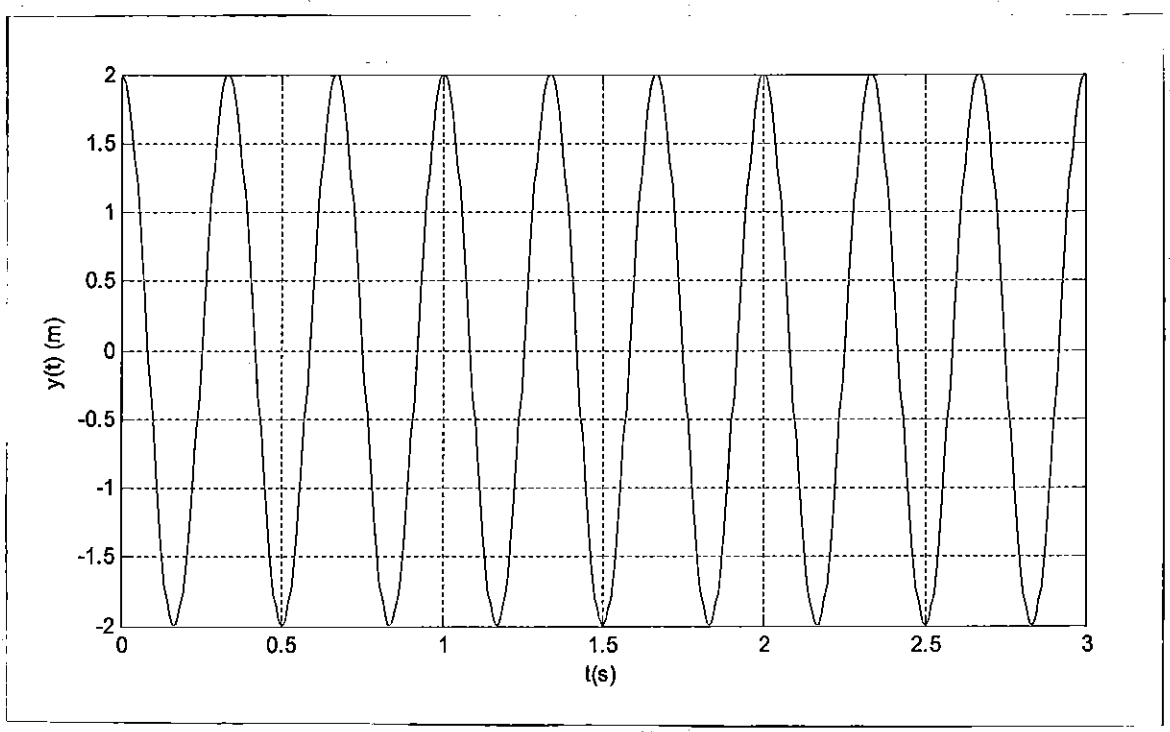
$$\Rightarrow \quad 6\pi t = \frac{3\pi}{2} - \pi/2 = \pi \longrightarrow \boxed{\therefore t = \frac{1}{6}\ \text{s.}}$$

(d) Plot the displacement $y(t)$ for $0 \le t \le 3$ s,

→ may plot using MATLAB w/ the fplot() function

MATLAB CODE:

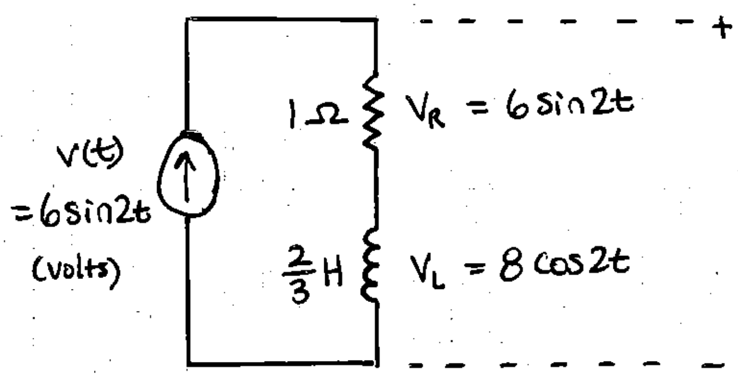
```
fplot('2*sin(6*pi*t + pi/2)', [0,3])
xlabel('t(s)')
ylabel('y(t) (m)')
grid on
```



Addition of Sinusoids:

The sum of a sine or cosine function of the same frequency is another sinusoid (sine or cosine) of that frequency.

EX: R-L circuit (EE2010 - Electric Circuits)



$V(t) = V_R + V_L$ volts

$$v(t) = V_R + V_L = 6\sin 2t + 8\cos 2t$$

Find: Write v(t) in terms of one sinusoid and a phase shift:

$$v(t) = 6\sin 2t + 8\cos 2t = M\sin(2t + \phi)$$

Thus, we really must find M and ϕ:

Solution:

Using the trigonometric identity,

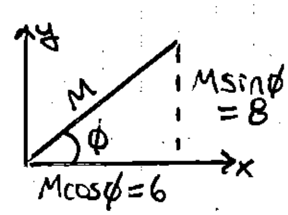
$$M\sin(2t+\phi) = M(\sin 2t\cos\phi + \cos 2t\sin\phi)$$

$$v(t) = 6\underbrace{\sin 2t}_{①} + 8\underbrace{\cos 2t}_{②} = M(\underbrace{\sin 2t}_{①}\cos\phi + \underbrace{\cos 2t}_{②}\sin\phi)$$

Equating coefficients on sin 2t and cos 2t on both sides:

① $\sin 2t$: $\quad 6 = M\cos\phi \quad \rightarrow$ (x-component)

② $\cos 2t$: $\quad 8 = M\sin\phi \quad \rightarrow$ (y-component)



$$\Rightarrow M = \sqrt{8^2 + 6^2} = 10, \quad \phi = \text{atan2}(8/6) = 53.13^\circ$$

$$\boxed{v(t) = 6\sin 2t + 8\cos 2t = 10\sin(2t + 53.13^\circ) \text{ volts}}$$

<u>NOTE</u>: we can also represent the same $v(t)$ as a <u>cosine</u>:

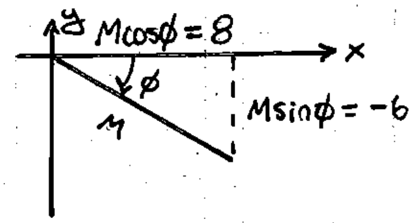
$$v(t) = 6\sin 2t + 8\cos 2t = M\cos(2t+\phi)$$

Recall now another identity,

$$M\cos(2t+\phi) = M(\cos 2t\cos\phi - \sin 2t\sin\phi)$$

$$v(t) = 6\underbrace{\sin 2t}_{①} + 8\underbrace{\cos 2t}_{②} = (M\cos\phi)\underbrace{\cos 2t}_{②} + (-M\sin\phi)\underbrace{\sin 2t}_{①}$$

Equating coefficients on both sides,



① $\sin 2t$: $\quad 6 = -M\sin\phi$

② $\cos 2t$: $\quad 8 = M\cos\phi$

$$\therefore M = \sqrt{8^2+6^2} = 10, \quad \phi = \operatorname{atan2}(-6/8) = -36.86^\circ$$

$$\boxed{\therefore v(t) = 6\sin 2t + 8\cos 2t = 10\cos(2t - 36.86^\circ) \text{ volts}}$$

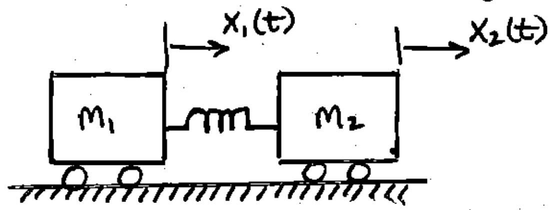
<u>NOTE</u>: in general, $\cos\theta = \sin(\theta + 90^\circ)$

$$\cos(\underbrace{2t - 36.86^\circ}_{\theta}) = \sin((2t - 36.86^\circ) + 90) = \sin(2t + 53.13^\circ)$$

Also in general:

$$\boxed{\begin{aligned} A\cos\omega t + B\sin\omega t &= \sqrt{A^2+B^2}\cos(\omega t - \operatorname{atan2}(B/A)) \\ A\cos\omega t + B\sin\omega t &= \sqrt{A^2+B^2}\sin(\omega t + \operatorname{atan2}(A/B)) \end{aligned}}$$

<u>EX</u>: Spring-mass system with two oscillating masses connected by a spring:



$$x_1(t) = 10\sin(2\pi t) \quad \text{\&} \quad x_2(t) = 5\sqrt{2}\cos(2\pi t + \tfrac{\pi}{4})$$

<u>Find</u>: the elongation of the spring given by $\delta(t) = x_2(t) - x_1(t)$

write $\delta(t)$ in the form $\delta(t) = M\cos(2\pi t + \phi)$

<u>Solution</u>: here we must use a trig. identity on both $x_2(t)$ & $\delta(t)$

$$\delta(t) = x_2(t) - x_1(t)$$

$$M\cos(2\pi t + \phi) = 5\sqrt{2}\cos(2\pi t + \tfrac{\pi}{4}) - 10\sin(2\pi t)$$

must seperate to get rid of phase

no need to modify, no phase!

We use the identity, $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$$M[\cos(2\pi t)\cos\phi - \sin(2\pi t)\sin\phi] =$$

$$= 5\sqrt{2}[\cos(2\pi t)\cancelto{\frac{\sqrt{2}}{2}}{\cos\tfrac{\pi}{4}} - \sin(2\pi t)\cancelto{\frac{\sqrt{2}}{2}}{\sin\tfrac{\pi}{4}}] - 10\sin(2\pi t)$$

Simplifying,

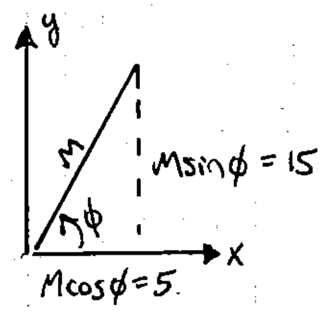
$$M[\underbrace{\cos(2\pi t)}_{①}\cos\phi - \underbrace{\sin(2\pi t)}_{②}\sin\phi] = 5\underbrace{\cos 2\pi t}_{①} - 5\underbrace{\sin 2\pi t}_{②} - 10\underbrace{\sin(2\pi t)}_{②}$$

Equating coefficients on $\cos(2\pi t)$ and $\sin(2\pi t)$,

① $\underline{\cos 2\pi t}$: $M\cos\phi = 5$

② $\underline{\sin 2\pi t}$: $-M\sin\phi = -5 - 10 = -15$

$\Rightarrow M\sin\phi = 15$



$$M = \sqrt{5^2 + 15^2} = 15.81$$

$$\phi = \text{atan2}(15/5) = 71.57°$$

$$\boxed{\therefore\ \delta(t) = x_2(t) - x_1(t) = 15.81\cos(2\pi t + 71.57°)}$$