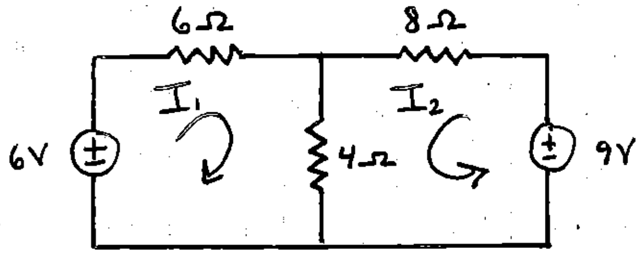
Systems of Equations in Engineering

<u>EX.</u>: Electric Circuits (EE 2010) – Two-Loop Circuit



... Two-Loop circuit

From Kirchoff's Voltage Law (KVL),

$$10I_1 + 4I_2 = 6 \quad \ldots (1)$$
$$4I_1 + 12I_2 = 9 \quad \ldots (2)$$

The above is a 2x2 system of equations for I_1 and I_2,

<u>Find</u>: I_1 and I_2

<u>Solution</u>: 4 possible solution methods

1.) Substitution Method
2.) Graphical Method
3.) Matrix Algebra
4.) Cramer's Rule

1.) <u>Substitution</u>: solve equation (1) for I_1 then plug into equation (2)

eqn (1): $10I_1 + 4I_2 = 6$

$$10I_1 = 6 - 4I_2$$

$$I_1 = \frac{6-4I_2}{10} \quad \ldots (1)$$

eqn (2): $4I_1 + 12I_2 = 9$

TOPS 35500

Now substituting (1) → (2),

$$\Rightarrow \quad 4\left(\frac{6-4I_2}{10}\right) + 12I_2 = 9$$

Solving for I_2,

$$\Rightarrow \quad 2.4 - 1.6I_2 + 12I_2 = 9$$

$$\Rightarrow \quad 10.4\,I_2 = 6.6 \quad \Rightarrow \quad I_2 = \frac{6.6}{10.4} = 0.6346\text{ A}$$

From (1),

$$I_1 = \frac{6-4I_2}{10} = \frac{6-4(0.6346)}{10} = 0.3462\text{ A}$$

$$\boxed{\therefore I_1 = 0.3462\text{ A} \;;\; I_2 = 0.6346\text{ A}}$$

2.) Graphical Method:

From eqn (1), $10I_1 + 4I_2 = 6$

$$\Rightarrow \quad 4I_2 = -10I_1 + 6$$

$$\Rightarrow \quad \boxed{I_2 = -\frac{5}{2}I_1 + \frac{3}{2}} \quad \ldots (1)$$

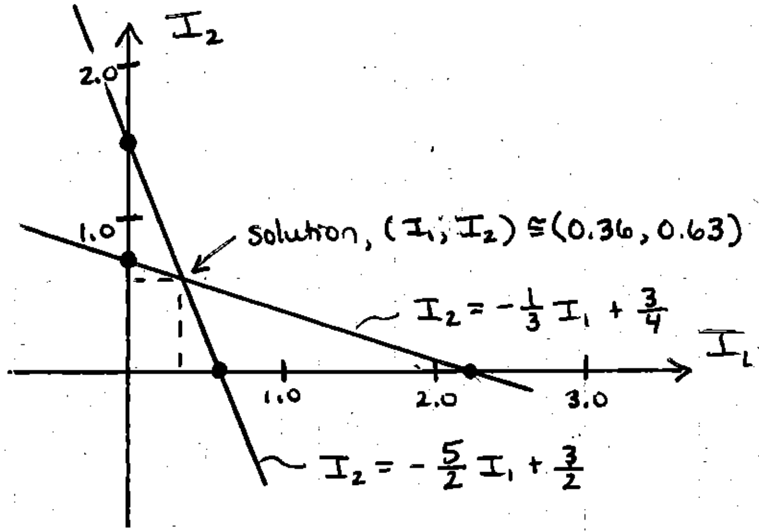
Similarly, from (2), $4I_1 + 12I_2 = 9$

$$\Rightarrow \quad 12I_2 = 9 - 4I_1$$

$$\Rightarrow \quad \boxed{I_2 = -\frac{1}{3}I_1 + \frac{3}{4}} \quad \ldots (2)$$

The above equations (1) and (2) are simply linear equations of the standard form: $y = mx + b$

The simultaneous solution of (1) and (2) is the intersection point of the two lines:



3.) Matrix Algebra:

2x2 system,

$$10 I_1 + 4 I_2 = 6 \quad \ldots (1)$$
$$4 I_1 + 12 I_2 = 9 \quad \ldots (2)$$

In matrix form,

$$\begin{bmatrix} 10 & 4 \\ 4 & 12 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 9 \end{Bmatrix}$$

OR,

$$\underline{A} \; \underset{\sim}{x} = \underset{\sim}{b}$$

where,

$$\underline{A} = \begin{bmatrix} 10 & 4 \\ 4 & 12 \end{bmatrix} \quad \text{... coefficient matrix}$$

$$\underset{\sim}{x} = \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} \quad \text{... vector of unknowns}$$

$$\underset{\sim}{b} = \begin{Bmatrix} 6 \\ 9 \end{Bmatrix} \quad \text{... right hand side (RHS)}$$

TOPS 35500

For any system $(n \times n)$: $\quad \underline{A}\,\underset{\sim}{x} = \underset{\sim}{b}$,

the solution is $\quad \underset{\sim}{x} = \underline{A}^{-1}\,\underset{\sim}{b}$

where $\quad A^{-1}$ ... inverse of the matrix $\underline{A}$

(This works for any size system).

For <u>2x2 system</u>: let $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $\quad \underline{A}^{-1} = \dfrac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

where $\quad \Delta = |\underline{A}| = \det(\underline{A})$ ... determinant of $\underline{A}$

$$\Delta = |\underline{A}| = \det(\underline{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

> <u>NOTE</u>: if $\Delta = |\underline{A}| = 0$, $\underline{A}^{-1}$ <u>does not exist</u> (singular)
>
> $\rightarrow$ this means <u>no solution</u> to $\underline{A}\,\underset{\sim}{x} = \underset{\sim}{b}$

<u>Back to the problem</u>: $\quad \underline{A} = \begin{bmatrix} 10 & 4 \\ 4 & 12 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\Rightarrow \underline{A}^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} 12 & -4 \\ -4 & 10 \end{bmatrix}$$

$$\Rightarrow \Delta = |\underline{A}| = ad - cb = (10)(12) - (4)(4) = 104$$

$$\Rightarrow \underline{A}^{-1} = \frac{1}{104}\begin{bmatrix} 12 & -4 \\ -4 & 10 \end{bmatrix} = \begin{bmatrix} 3/26 & -1/26 \\ -1/26 & 5/52 \end{bmatrix}$$

Solution: $\underset{\sim}{X} = \underline{A}^{-1}\underset{\sim}{b} = \begin{bmatrix} 3/26 & -1/26 \\ -1/26 & 5/52 \end{bmatrix}\begin{Bmatrix} 6 \\ 9 \end{Bmatrix}$

$$\underset{\sim}{X} = \begin{Bmatrix} (3/26)(6) + (-1/26)(9) \\ (-1/26)(6) + (5/52)(9) \end{Bmatrix} = \begin{Bmatrix} \frac{18-9}{26} \\ \frac{-12+45}{52} \end{Bmatrix} = \begin{Bmatrix} 0.3462 \\ 0.6346 \end{Bmatrix}$$

$$\boxed{\therefore \underset{\sim}{X} = \begin{Bmatrix} 0.3462 \\ 0.6346 \end{Bmatrix} \text{Amps} = \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix}}$$

4.) Cramer's Rule:

For any system $\underline{A}\underset{\sim}{X} = \underset{\sim}{b}$

$$X_1 = \frac{|\underline{A}_1|}{|\underline{A}|}, \quad X_2 = \frac{|\underline{A}_2|}{|\underline{A}|}, \quad \ldots, \quad X_i = \frac{|\underline{A}_i|}{|\underline{A}|}$$

where $\underline{A}_i$ is obtained by replacing the i^{th} column of $\underline{A}$ with the vector $\underset{\sim}{b}$.

For a 2x2 system,
$$a_{11}X_1 + a_{12}X_2 = b_1$$
$$a_{21}X_1 + a_{22}X_2 = b_2$$

Matrix Form: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix}$

Cramer's Rule:

$$X_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad X_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Back to the problem, $\begin{bmatrix} 10 & 4 \\ 4 & 12 \end{bmatrix}\begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 9 \end{Bmatrix}$

By Cramer's Rule,

$$I_1 = \frac{\begin{vmatrix} 6 & 4 \\ 9 & 12 \end{vmatrix}}{\begin{vmatrix} 10 & 4 \\ 4 & 12 \end{vmatrix}} = \frac{6(12) - 9(4)}{10(12) - 4(4)} = \frac{36}{104} = 0.3462$$

$$I_2 = \frac{\begin{vmatrix} 10 & 6 \\ 4 & 9 \end{vmatrix}}{104} = \frac{10(9) - 4(6)}{104} = \frac{66}{104} = 0.6346$$

$$\Rightarrow \quad \boxed{I_1 = 0.3462 \text{ A}, \quad I_2 = 0.6346 \text{ A}}$$

<u>NOTE</u>: Cramer's Rule is probably the fastest method for 2x2 systems.

However, not faster than MATLAB:

CODE:

```
A = [10  4 ; 4  12]
b = [6 ; 9]
```

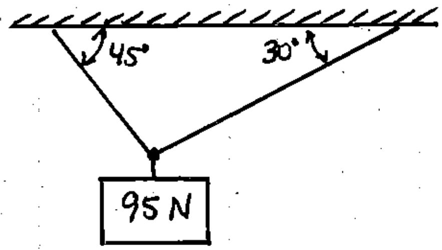
$$x = \text{inv}(A) * b = \begin{Bmatrix} 0.3462 \\ 0.6346 \end{Bmatrix}$$

<u>OR</u>:

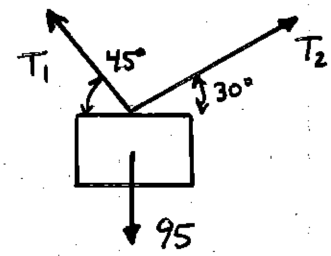
$$x = A \backslash b = \begin{Bmatrix} 0.3462 \\ 0.6346 \end{Bmatrix}$$

"left division" → much faster for large systems than inv(A)*b

EX: ME2120 - Statics



FBD:



$\Sigma F_x = 0$: $\quad -T_1 \cos 45° + T_2 \cos 30° = 0$

$$-0.7071 T_1 + 0.866 T_2 = 0 \qquad \ldots (1)$$

$\Sigma F_y = 0$: $\quad T_1 \sin 45° + T_2 \sin 30° = 95$

$$0.7071 T_1 + 0.5 T_2 = 95 \qquad \ldots (2)$$

1.) Substitution: from (1), $0.866 T_2 = 0.7071 T_1$

$$\Rightarrow T_2 = 0.8165 \, T_1$$

plugging into (2): $0.7071 T_1 + 0.5(0.8165 T_1) = 95$

$$\Rightarrow 1.115 T_1 = 95 \qquad \therefore T_1 = 85.17 \text{ N}$$

$$T_2 = 0.8165(85.17) = 69.55 \text{ N}$$

$$\boxed{\therefore T_1 = 85.17 \text{ N}, \; T_2 = 69.55 \text{ N}}$$

2.) Matrix Inverse:
$$-0.7071 T_1 + 0.866 T_2 = 0$$
$$0.7071 T_1 + 0.5 T_2 = 95$$

In matrix form,
$$\underbrace{\begin{bmatrix} -0.7071 & 0.866 \\ 0.7071 & 0.5 \end{bmatrix}}_{\underset{\sim}{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}} \underbrace{\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}}_{\underset{\sim}{x}} = \underbrace{\begin{Bmatrix} 0 \\ 95 \end{Bmatrix}}_{\underset{\sim}{b}}$$

$$\underset{\sim}{A}^{-1} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Delta = (-.7071)(0.5) - (0.7071)(0.866)$$
$$= -0.9659$$

$$\underline{A}^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-0.9659}\begin{bmatrix} 0.5 & -0.866 \\ -0.7071 & -0.7071 \end{bmatrix}$$

$$\underline{A}^{-1} = \begin{bmatrix} -0.5177 & 0.8966 \\ 0.7321 & 0.7321 \end{bmatrix}$$

$$\underset{\sim}{x} = \underline{A}^{-1}\underset{\sim}{b} \Rightarrow \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{bmatrix} -0.5177 & 0.8966 \\ 0.7321 & 0.7321 \end{bmatrix}\begin{Bmatrix} 0 \\ 95 \end{Bmatrix}$$

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 0 + 0.8966(95) \\ 0 + 0.7321(95) \end{Bmatrix} = \begin{Bmatrix} 85.2 \\ 69.5 \end{Bmatrix} N$$

3.) <u>Cramer's Rule</u>:

$$T_1 = \frac{\begin{vmatrix} 0 & 0.866 \\ 95 & 0.5 \end{vmatrix}}{-0.9659} = \frac{0 - 95(0.866)}{-0.9659} = 85.2$$

$$T_2 = \frac{\begin{vmatrix} -0.7071 & 0 \\ 0.7071 & 95 \end{vmatrix}}{-0.9659} = \frac{(-0.7071)(95) - 0}{-0.9659} = 69.5$$

$$\boxed{\therefore\ T_1 = 85.2\ N\ ,\ T_2 = 69.5\ N}$$