Derivatives in Engineering

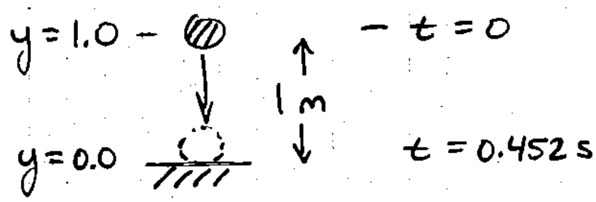
The Derivative:

What is it? Why do engineers need to know it?

EXAMPLE: Velocity and Acceleration

Consider a ball dropped from a height $y = 1.0$ m.

A student measures the time to impact as $t = 0.452$ s.



An engineer poses the following questions:

1.) What is the average velocity of the ball?

2.) What is the speed at impact?

3.) How fast is the ball accelerating?

Solutions: 1.) average velocity, $\bar{V}$: $\bar{V} = \dfrac{\text{total distance}}{\text{total time}} = \dfrac{\Delta y}{\Delta t}$

$$\Rightarrow \bar{V} = \frac{-1.0}{0.452} = -2.21 \ \frac{m}{s}$$

2.) speed at impact:

NOT ENOUGH INFO!

The student proposes to collect more data and obtains,

t (s)	0	0.1	0.2	0.3	0.4	0.452
y(t) (m)	1.0	0.951	0.804	0.559	0.215	0

He then proposes to calculate the average velocity in each interval: $\bar{V} = \frac{\Delta y}{\Delta t}$

eg. for the interval $t = [0, 0.1]$

$$\bar{V} = \frac{0.951 - 1.0}{0.1 - 0.0} = -0.490 \ \frac{m}{s}$$

The results are as follows:

interval	[0,0.1]	[0.1,0.2]	[0.2,0.3]	[0.3,0.4]	[0.4,0.452]
$\bar{V}$ (m/s)	-0.490	-1.47	-2.45	-3.44	-4.13

The student then proposes that the approximate answer to 2.) as -4.13 m/s, but claims he would need an ∞ # of data points to get it exactly right!

$$\text{ie,} \quad v(t = 0.452) = \lim_{t \to 0.452} \frac{y(0.452) - y(t)}{0.452 - t}$$

The engineer suggests that this looks alot like the definition of a derivative,

$$\text{ie} \quad v(t) = \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} \equiv \frac{dy}{dt}$$

$$\text{where} \quad \Delta t = 0.452 - t$$

The engineer suggest a quadratic curve fit of the measured data, which gives:

$$y(t) = 1.0 - 4.905\, t^2 \quad \text{(meters)}$$

The velocity at any time is thus

$$v(t) = \frac{dy}{dt} = \frac{d}{dt}\left(1.0 - 4.905t^2\right) = -9.81t \quad \left(\frac{m}{s}\right)$$

At impact: $t = 0.452$ sec

$$\Rightarrow v = -9.81(0.452) = -\underline{4.43} \ \frac{m}{s}$$

(<u>NOTE</u>: the approx. of the last interval gave -4.13 m/s)

3.) <u>Acceleration</u>:

Without taking more data, the engineer now recalls that acceleration is the <u>rate of change</u> of velocity:

$$a(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \equiv \frac{dv}{dt} \equiv \frac{d^2y}{dt^2}$$

$$\Rightarrow a(t) = \frac{d}{dt}(-9.81t) = \underline{-9.81} \ \frac{m}{s^2}$$

(<u>NOTE</u>: acceleration due to gravity is <u>constant</u>!)

Suppose now that the ball is thrown upwards with an initial velocity of $V_0 = 4.43$ m/s.

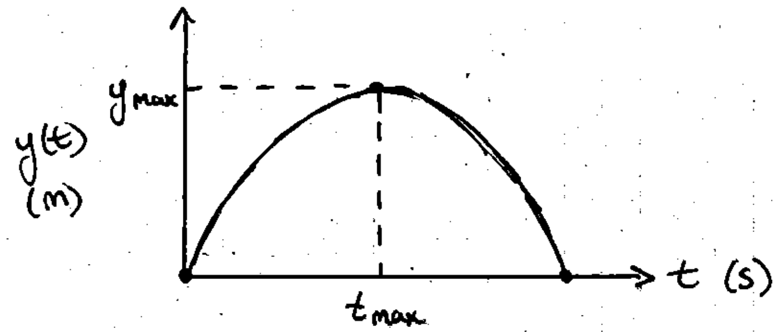
What is the <u>maximum</u> height y_{max}?

How long does it take to get there?

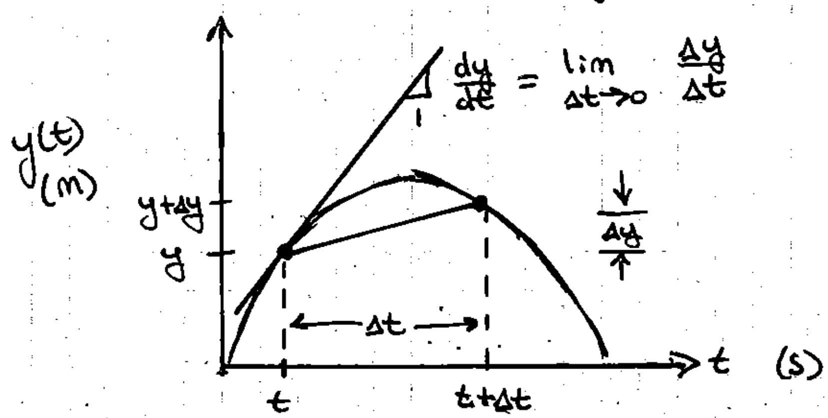
What is the velocity at y_{max}?

The engineer suggests that the height is given by the governing equation: $y(t) = 4.43t - 4.905t^2$ (m)

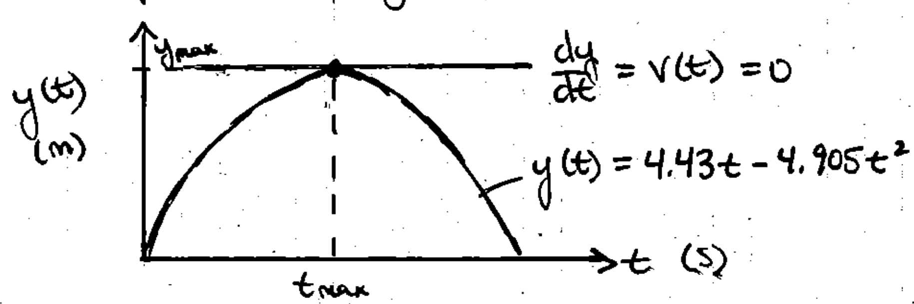
<u>Governing Equation</u>: $y(t) = 4.43t - 4.905t^2$ (m)



Based on the definition of the derivative, the velocity v(t) at any time t is the slope of the <u>tangent line</u> to y(t):



At y_{max}, the slope of the tangent line is <u>zero</u>:



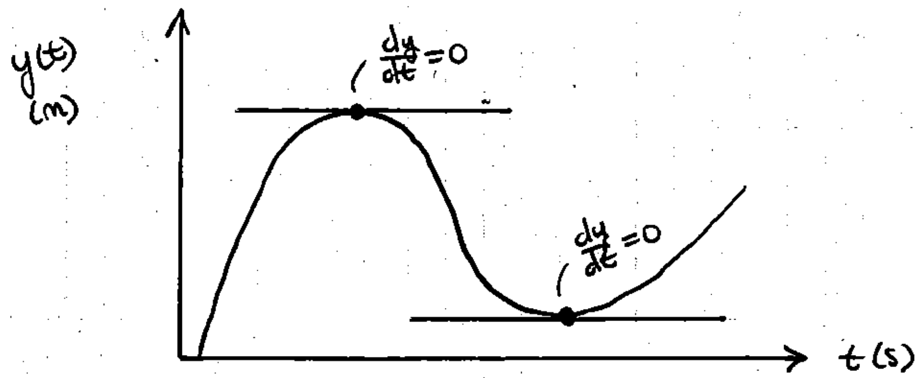
$v(t) = \frac{dy}{dt} = 4.43 - 9.81t$

at $t = t_{max}$, $v = \frac{dy}{dt} = 0 \Rightarrow 4.43 - 9.81t = 0$

$\boxed{\therefore t_{max} = 0.452 \text{ s}}$ ... looks familiar?

$y_{max} = y(0.452) = 4.43(0.452) - 4.905(0.452)^2 = \boxed{1.0 \text{ m} = y_{max}}$

Aside: without plotting, how can you tell this is a maximum or minimum?



Maximum: $\frac{dy}{dt} = 0$; $\frac{d^2y}{dt^2} < 0$ (concave down)

Minimum: $\frac{dy}{dt} = 0$; $\frac{d^2y}{dt^2} > 0$ (concave up)

Back to ball problem,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(4.43 - 9.81t) = -9.81 < 0$$ ... maximum!