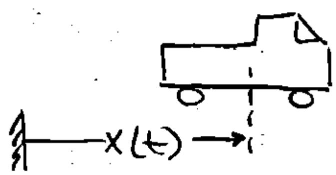
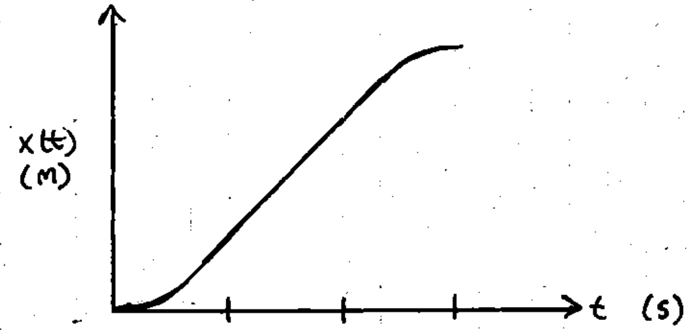
Derivatives in Dynamics

Application of Derivatives in Dynamics:

Position, Velocity, and Acceleration (PHY2400 ; ME2210)

→ The motion of an object is defined by its position, $x(t)$:

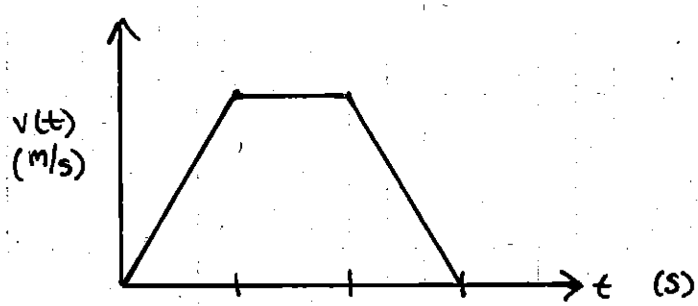




→ The velocity $v(t)$ is the instantaneous rate of change of the position (ie, the derivatives):

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \equiv \frac{dx}{dt}$$

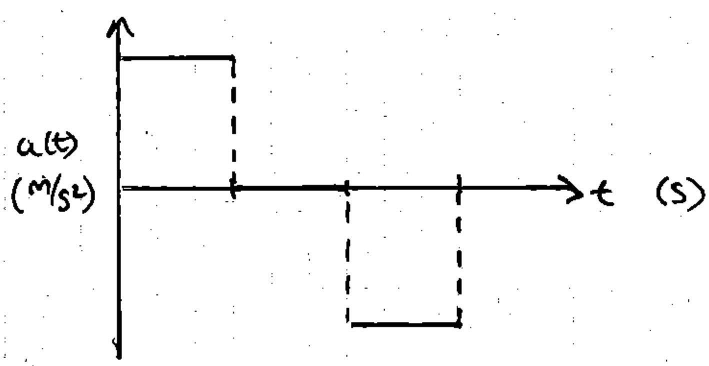
(slope of $x(t)$)



→ The acceleration $a(t)$ is the instantaneous rate of change of the velocity:

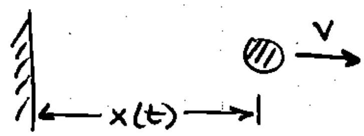
$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

(slope of $v(t)$)



EXAMPLE: Velocity & Acceleration: (PHY 2400 & ME 2210)

The motion of a particle is defined by its position, $x(t)$ in m:



Determine the position, velocity, and acceleration at $t = 0.5\,s$ for:

(a) $x(t) = \sin 2\pi t$ m

(b) $x(t) = 3t^3 - 4t^2 + 2t + 6$ m

(c) $x(t) = 20\cos(3\pi t) - 5t^2$ m

Solution:

(a) $x(t) = \sin 2\pi t$ m

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}\left(\sin 2\pi t\right) = 2\pi\cos(2\pi t) \ \frac{m}{s}$$

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(2\pi\cos 2\pi t\right) = -4\pi^2 \sin 2\pi t \ \frac{m}{s^2}$$

Now evaluating at $t = 0.5\,s$:

$$x(0.5) = \sin(2\pi(0.5)) = \sin\pi = \boxed{0 \text{ m} = x(0.5)}$$

$$v(0.5) = 2\pi\cos(2\pi(0.5)) = 2\pi\cos\pi = \boxed{-2\pi \text{ m/s} = v(0.5)}$$

$$a(0.5) = -4\pi^2 \sin(2\pi(0.5)) = -4\pi^2\sin\pi = \boxed{0 \text{ m/s}^2 = a(0.5)}$$

(b) $x(t) = 3t^3 - 4t^2 + 2t + 6$ m

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}\left(3t^3 - 4t^2 + 2t + 6\right)$$

$$= 3(3)t^2 - 4(2)t + 2(1) + 0 = 9t^2 - 8t + 2 \ \frac{m}{s}$$

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(9t^2 - 8t + 2\right) = 9(2)t - 8(1) + 0$$

$$= 18t - 8 \ \text{m/s}^2$$

Again, evaluating at $t = 0.5\,s$:

$$x(0.5) = 3(0.5)^3 - 4(0.5)^2 + 2(0.5) + 6 = \boxed{6.375\ m = x(0.5)}$$

$$v(0.5) = 9(0.5)^2 - 8(0.5) + 2 = \boxed{0.25\ m/s = v(0.5)}$$

$$a(0.5) = 18(0.5) - 8 = \boxed{1.0\ m/s^2 = a(0.5)}$$

(c) $x(t) = 20\cos 3\pi t - 5t^2$ m

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}\left(20\cos 3\pi t - 5t^2\right) = -60\pi \sin 3\pi t - 10t \ \ m/s$$

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(-60\pi \sin 3\pi t - 10t\right) = -180\pi^2 \cos 3\pi t - 10 \ \ m/s^2$$

Evaluating at $t = 0.5\,s$:

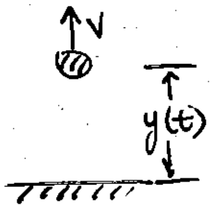
$$x(0.5) = 20\cos(3\pi(0.5)) - 5(0.5)^2 = \boxed{-1.25\ m = x(0.5)}$$

$$v(0.5) = -60\pi \sin(3\pi(0.5)) - 10(0.5) = \boxed{183.5\ m/s = v(0.5)}$$

$$a(0.5) = -180\pi^2 \cos(3\pi(0.5)) - 10 = \boxed{-10\ m/s^2 = a(0.5)}$$

EXAMPLE: Velocity & Acceleration: (PHY2400 & ME2210)

The motion of a particle in the vertical plane is defined by its position, $y(t)$ in meters:



$$y(t) = \frac{1}{3}t^3 - 5t^2 + 21t + 10 \text{ m}$$

Determine the values of the position and acceleration when the velocity is zero.

Solution: $y(t) = \frac{1}{3}t^3 - 5t^2 + 21t + 10$ m

$$v(t) = \frac{dy}{dt} = t^2 - 10t + 21 \quad \text{m/s}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2} = 2t - 10 \quad \frac{\text{m}}{\text{s}^2}$$

zero velocity: $v(t) = t^2 - 10t + 21 = 0$

Solving via factoring: $v(t) = (t-3)(t+7) = 0$

$$\boxed{\Rightarrow t = 3\text{s} \;\; \& \;\; t = 7\text{s}}$$

*NOTE: could also use quadratic formula:

$$t = \frac{10 \pm \sqrt{(10)^2 - 4(1)(21)}}{2(1)} = \frac{10 \pm \sqrt{16}}{2} = 5 \pm 2$$

$$\therefore t = 3\text{s} \;\; \& \;\; t = 7\text{s}$$

@ t = 3: $y(3) = \frac{1}{3}(3)^3 - 5(3)^2 + 21(3) + 10 = \boxed{37\ m = y(3)}$

$a(3) = 2(3) - 10 = \boxed{-4\ m/s^2 = a(3)}$

@ t = 7: $y(7) = \frac{1}{3}(7)^3 - 5(7)^2 + 21(7) + 10 = \boxed{26.3\ m = y(7)}$

$a(7) = 2(7) - 10 = \boxed{4\ m/s^2 = a(7)}$

NOTE: above information can be used to sketch a graph of $y(t)$:

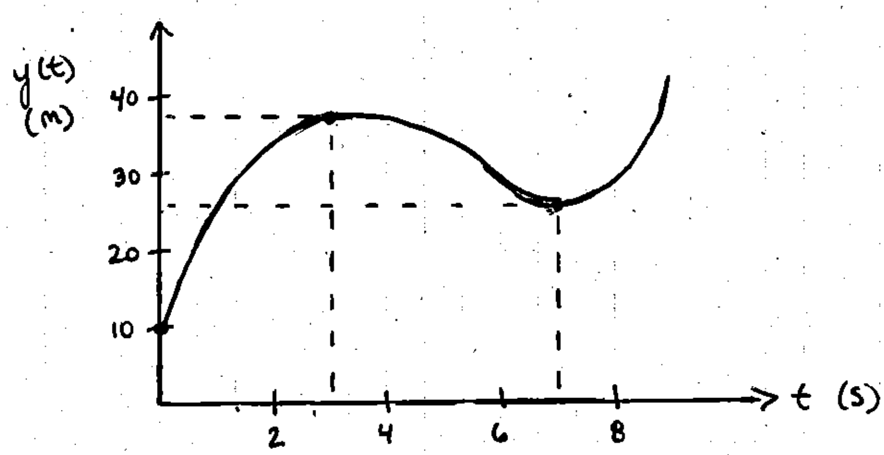
$$y(t) = \frac{1}{3}t^3 - 5t^2 + 21t + 10 \quad m$$

- at $t = 0$, $y = 10\ m$
- at $t = 3$, $y = 37\ m$, $v = \frac{dy}{dt} = 0\ m/s$, $a = \frac{d^2y}{dt^2} = -4\ \frac{m}{s^2} < 0$

  $\therefore$ at $t = 3$, we have a local maximum

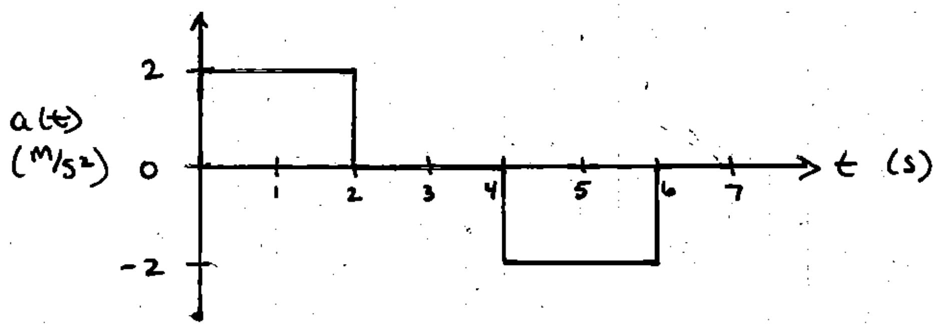
- at $t = 7$, $y = 26.3\ m$, $V = \frac{dy}{dt} = 0\ m/s$, $a = \frac{d^2y}{dt^2} = 4\ m/s^2 > 0$

  $\therefore$ at $t = 7$, we have a local minimum



EXAMPLE: Velocity & Acceleration: (PHY2400 & ME2210)

The acceleration of a particle is measured as follows:



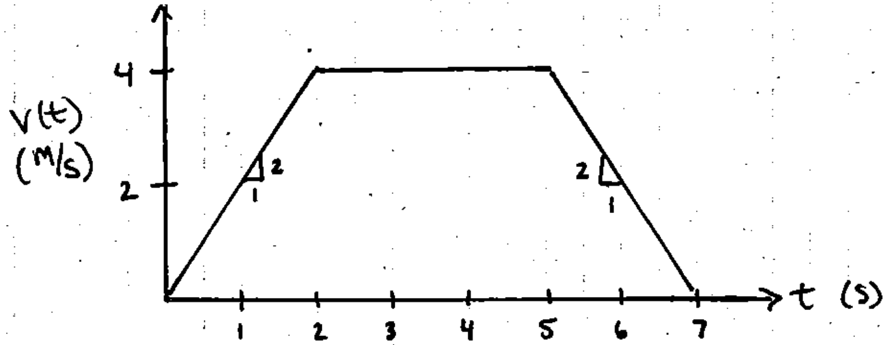
Knowing that the particle starts from rest and travels a total of 16 m, sketch graphs of the position $x(t)$ and velocity $v(t)$.

Solution: Begin w/ $v(t)$, knowing that $v(0) = 0$ and the slope of $v(t)$ is $a(t)$: $\frac{dv}{dt} = a(t)$

$0 \leq t \leq 2$ s: $a(t) = \frac{dv}{dt} = 2$ (constant slope)

$2 \leq t \leq 4$ s: $a(t) = \frac{dv}{dt} = 0$ ($v(t)$ is constant)

$4 \leq t \leq 6$ s: $a(t) = \frac{dv}{dt} = -2$ (constant slope)



Now we can use graph of $v(t)$ to construct $x(t)$ knowing that the slope of $x(t)$ is $v(t)$: $\frac{dx}{dt} = v(t)$.

$0 \le t \le 2$ s: $v(t)$ is a straight line ; $v(t) = \frac{dx}{dt} = 2t$

from the derivative handout, $x(t) = t^2 + C$

(ie. $\frac{dx}{dt} = \frac{d}{dt}(t^2 + C) = 2t$ ✓)

since $x(0) = 0$, $C = 0$

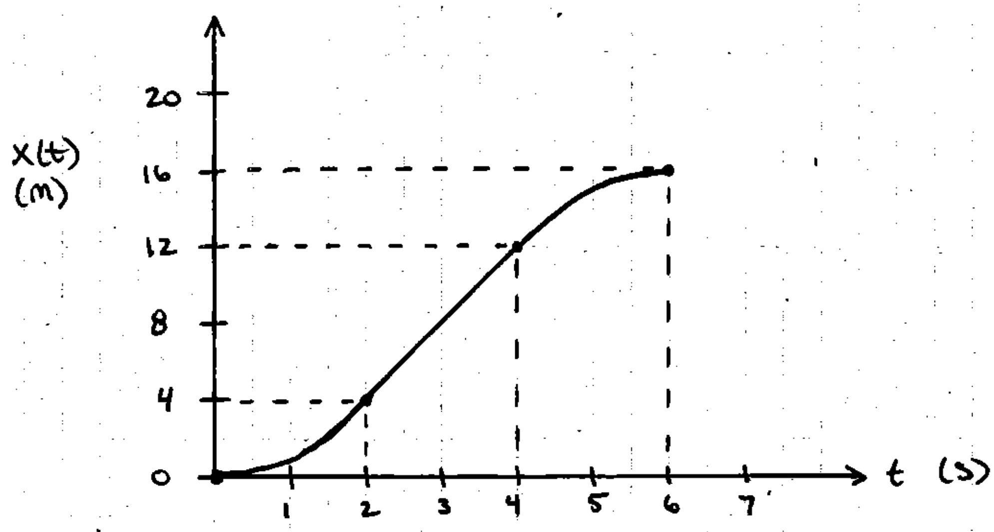
$\Rightarrow$ $x(t) = t^2$, $x(0) = 0$, $x(2) = 4$ m

$2 \le t \le 4$ s: $v(t) = \frac{dx}{dt} = 4$ (constant slope)

$\Rightarrow$ $x(t)$ is a straight line w/ slope of 4

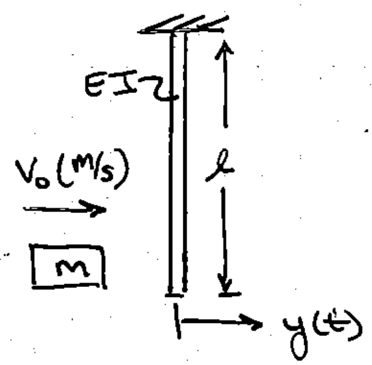
$4 \le t \le 6$ s: $v(t)$ is a straight line that is decreasing

$\Rightarrow$ $x(t)$ is a quadratic w/ decreasing slope, and ending at $x = 16$ m (given) w/ zero slope (from $v(t)$)



EXAMPLE: Velocity & Acceleration (ME 4140 & ME 4600)

A mass m impacts a cantilever with velocity V_0 :



ℓ... length (m)

EI... flexural rigidity ($N \cdot m^2$)

The resulting displacement of the cantilever is $y(t)$,

$$y(t) = \frac{V_0}{\omega} \sin \omega t, \quad \text{where} \quad \omega = \sqrt{\frac{3EI}{m\ell^3}}$$

Find the following,

(a) the maximum displacement, y_{max}

(b) the values of the displacement and acceleration when the velocity is zero

Solution:

(a) y_{max} occurs when $\sin \omega t = 1$

$$\Rightarrow y_{max} = \frac{V_0}{\omega} = \frac{V_0}{\sqrt{3EI/m\ell^3}} = V_0 \sqrt{\frac{m\ell^3}{3EI}} \text{ m}$$

(b) velocity, $v(t) = \frac{dy}{dt} = \frac{d}{dt}\left(\frac{V_0}{\omega} \sin \omega t\right)$

$$\therefore v(t) = \frac{V_0}{\omega} \omega \cos \omega t = V_0 \cos \omega t \ \text{(m/s)}$$

velocity equal to zero: $v(t) = 0 = V_0 \cos \omega t$

occurs when $\Rightarrow \omega t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

Selecting, $\omega t = \frac{\pi}{2}$:

$$y\left(\frac{\pi}{2}\right) = \frac{V_0}{\omega}\sin\left(\frac{\pi}{2}\right) = \frac{V_0}{\omega} = y_{max} \text{ !!}$$

$\Rightarrow$ NOTE: $y(t)$ is maximum when $v(t) = \frac{dy}{dt} = 0$!

acceleration: $a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2}$

$$a(t) = \frac{d}{dt}(V_0 \cos\omega t) = -V_0\,\omega \sin\omega t$$

when $v(t) = 0$, $\omega t = \pi/2$,

$$a(t) = -V_0\,\omega\sin\omega t \Rightarrow a\left(\frac{\pi}{2}\right) = -V_0\,\omega\sin(\pi/2)$$

$$a\left(\frac{\pi}{2}\right) = -V_0\,\omega = -V_0\sqrt{\frac{3EI}{m\ell^3}}\ \frac{m}{s^2}$$

NOTE: $a(t) = \omega^2\left(\frac{V_0}{\omega}\sin\omega t\right) = -\omega^2 y(t)$

$\Rightarrow$ acceleration is maximum when displacement $y(t)$ is also maximum!