Laboratory 7

Differential Equations in Engineering: The Leaking Bucket

7.1 Laboratory Objective

The objective of this laboratory is to learn about first order differential equations and their application to a leaking bucket.

7.2 Educational Objectives

After performing this experiment, students should be able to:

1. Understand the modeling of a leaking bucket dynamic system.
2. Measure the key parameters of a leaking bucket dynamic system.
3. Validate a mathematical model of the leaking bucket with observed data.

7.3 Background

Differential equations are an integral part of engineering. Almost all system response can be described by a differential equation. Knowledge of how to solve these problems is key to an engineer's success. This lab looks at one classification of a differential equation; first order, constant coefficient, and homogeneous.

7.3.1 The Leaking Bucket

The system shown in Figure 7.1 can be described by investigating the behavior of the water. The following equation describes the volumetric flow rate, Q of the system.
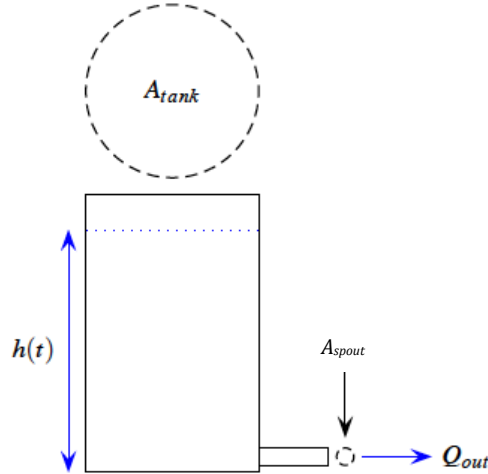
$$Q_{in} - Q_{out} - Q_{stored} = 0$$

Figure 7.1: Leaking Bucket

There will not be any water flowing into the system, therefore $Q_{in} = 0$.

$$Qstored = - Qout$$

The volumetric flow rate is found by multiplying the velocity by the area.

$$A_{tank} \frac{dh(t)}{dt} = A_{spout} v(t)$$

From fluids, the velocity of the water coming out of the straw is $\sqrt{2gh}$

$$A_{tank} \frac{dh(t)}{dt} = A_{spout} \sqrt{2gh(t)}$$

Rearranging terms and writing all constants as one:

$$A_{tank} \frac{dh(t)}{dt} - A_{spout} \sqrt{2g} \sqrt{h(t)} = 0$$

$$A_{tank} \frac{dh(t)}{dt} + K\sqrt{h(t)} = 0$$

$$A_{tank} \dot{h}(t) + K\sqrt{h(t)} = 0$$

The above equation cannot be solved using the methods of this class because the h on the second term of the equation is under a square root. To accommodate this, the governing equation that will be solved in this lab will be approximated without the square root:

$$A_{tank} \dot{h}(t) + K\sqrt{h(t)} \approx A_{tank} \dot{h}(t) + Kh(t) = 0$$

The solution to the governing equation is:

$$h(t) = ce^{-\left(\frac{1}{\tau}\right)t} = ce^{-\left(\frac{K}{A_{tank}}\right)t}$$

Where C is the initial height of the water and the system time constant is defined as $\tau = A_{tank}/K$.

7.4 Procedure

Follow the steps outlined below after the Lab Teaching Assistant has explained how to use the laboratory equipment.

1. Place a piece of tap axially along the bottle from the spout to halfway up the bottle.
2. Fill the bottle with water halfway. Make sure that the spout is closed so water does not leak out.
3. Place a mark on the tape to indicate the initial height of the water.
4. Open the valve and allow the water to flow out into the drain pan.
5. On the tape, mark the height of the water level every five seconds until the water level is approximately 2cm above the spout.
6. Remove tape and complete 7.1.

Table 7.1

Regular Spout			Spout with Extension		
Time t (sec)	Height h (m)	ln(h)	Time t (sec)	Height h (m)	ln(h)
0	h_1	ln(h_1)	0	h_1	ln(h_1)
5	h_2	ln(h_2)	5	h_2	ln(h_2)
10	h_3	ln(h_3)	10	h_3	ln(h_3)
etc.	etc.	etc.	etc.	etc.	etc.

7. Using Microsoft Excel, plot h vs. t for both spouts.
8. Using Microsoft Excel, plot ln(h) vs. t for both spouts.
   a. Fit a line to the data and place the equation on the plot.

NOTE: The slope of this straight line is $-K/A_{tank}$. The time constant τ is simply the negative inverse of the slope.

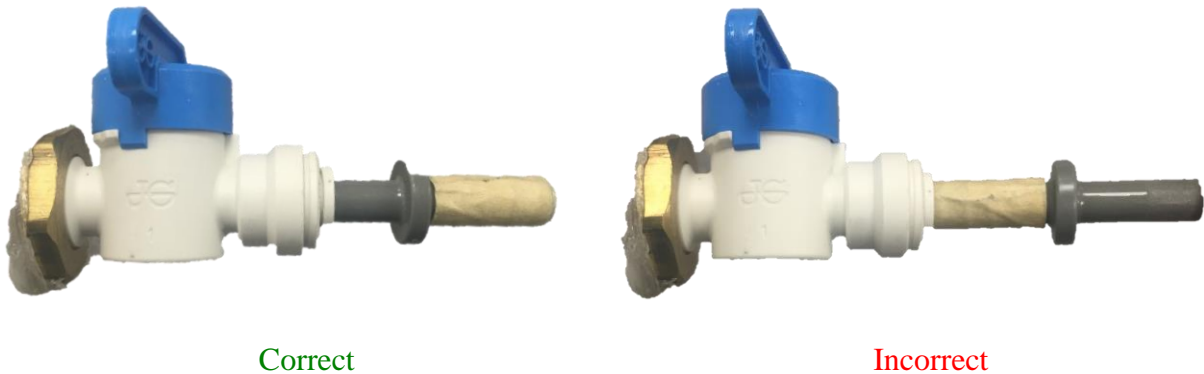<div align="center">Correct   Incorrect</div>

Figure 7.2: Correct and incorrect way of inserting spout.

7.5 Lab Requirements

1. Write an abstract for this lab and submit it to the Lab 7 folder in your lab section's abstract folder found in the Pilot Dropbox. (Required to pass course.)

2. Complete Table 7.1 in Microsoft Excel and insert after this page. (2 points)

3. Insert two plots of h vs. t after this page. (2 points each)

4. Insert two plots of $\ln(h)$ vs. t after this page. (2 points each)

5. Derive by hand the equation of the straight line for the "ln plot" in terms of C, K, and, $A_{tank.}$ (2 points)

   HINT: Start by taking the natural log of both sides of Equation 7.1 and algebraically simplify.

6. Answer the following questions:

a. What is the time constant for the regular spout? (Don't forget units.) (2 points)

b. What is the time constant for the spout with extension? (Don't forget units.) (2 points)

c. What type of energy is stored in the water? (2 points)